جمهورية العراق
وزارة التعليم العالي والبحث العلمي
الجامعة المستنصرية
كلية التربية قسم الرياضيات
الدراسات الصباحية والمسائية

الجبر الخطي

المرحلة الاولى

الجزء الثاني

1500

2-7 The Rank Of The Matrix رتبة المصفوفة

تعريف :-
لتكن A مصفوفة ذات سعة m*n فان رتبة المصفوفة A هي السعة لاكبر مصفوفة جزئية من A المحدد لها لا يساوي صفر .

مثال:- جد رتبة المصفوفة

Ex:-

$$A=\begin{pmatrix} 3 & 1 & 0 \\ 5 & 7 & 1 \end{pmatrix}$$

لا يوجد محدد لهذة المصفوفة لأنها ليست مربعة لذلك نستخرج منها مصفوفة جزئية 2*2

المصفوفة B محددها لا يساوي صفر

إذن رتبة A تساوي 2

$$B=\begin{pmatrix} 3 & 1 \\ 5 & 7 \end{pmatrix}_{2*2}$$

مثال:-

Ex:-

جد رتبة المصفوفة

$$A = \begin{pmatrix} 5 & 7 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 9 & 0 & 0 & 0 \end{pmatrix}_{3*4}$$

الحل :-

Sol :-

نستخرج مصفوفات جزئية ثم نجد المحدد لها فأذا كان المحدد لا يساوي صفر فأن رتبة A تساوي سعة تلك المصفوفة ...... أما أذا كان المحدد يساوي صفر نستخرج مصفوفات جزئية بسعة أصغر ثم نجد المحدد لها ومن خلالة نحدد رتبة المصفوفة

$$B = \begin{pmatrix} 5 & 7 & 2 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{pmatrix}_{3*3} \Longrightarrow |B| = 0$$

$$B=\begin{pmatrix} 5 & 7 \\ 9 & 0 \end{pmatrix}_{2*2}$$

$|B| = -63$

Then rank A = 2

Chapter Four

Vector Space فضاء المتجهات

S_1 The Vectors المتجهات

Def :- Let V be a ***vector*** in R^n. Then $V = (v_1, v_2, \ldots, v_n)$,where v_i (i=1,…,,n) are called the ***components*** of the vector V

ملاحظة(1) :-

كل متجه $\vec{V} = (v_1, v_2, \ldots, v_n)$ مرتبط بقطعة مستقيم متجهة بدايتها عند نقطة الاصل(***Origin***) O (0 , 0,…,0)

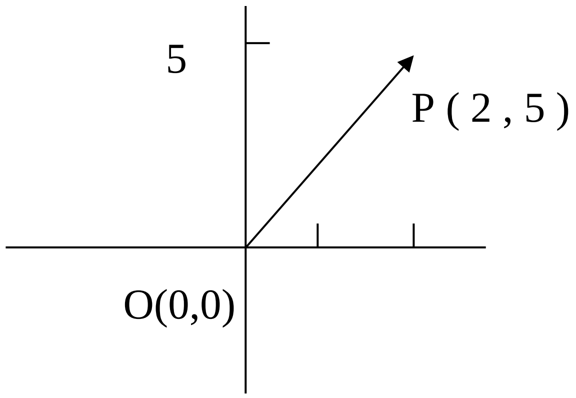
ونهايتها عند النقطة $P(v_1, v_2, \ldots, v_n)$ نرمز لقطعة المستقيم المتجهة من الصفر الى P بالرمز $\overrightarrow{OP}$

وان اتجاه قطعة المستقيم هو الزاوية المصنوعة مع محور X الموجب .

(1) Ex:-

V = (2 , 5)

Sol :-



0 = (0 , 0)   P = (2 , 5)

$\vec{V} = OP = (2, 5)$

(2) Ex:-

P= (-7 , 2)   0 = (0 , 0)

$\vec{V} = OP = (-7, 2)$

Definitions تعاريف :-

Def(1):- Let $V = (V_1, V_2, \ldots, V_n)$ and $W = (w_1, w_2, \ldots, w_n)$ are two vectors. Then $V = W$ if $v_i = w_i$, where, $i=1,\ldots,n$

Ex:-

$V = (5, 7)$, $W = (4, 2)$

$\vec{V} \neq \vec{W}$

Def(2):- Let $V = (V_1, V_2, \ldots, V_n)$ and $U = (U_1, U_2, \ldots, U_n)$ are two vectors. Then

$\vec{U} + \vec{V} = (U_1, U_2, \ldots, U_n) + (V_1, V_2, \ldots, V_n)$
$= (U_1 + V_1, U_2 + V_2, \ldots, U_n + V_n)$.

And

$U - V = U + (-V)$

$= (U_1 - V_1, U_2 - V_2, \ldots, U_n - V_n)$.

Ex:- Let $V = (5, 4)$, $W = (3, -2)$ then find $V + W$

Sol :-

$\vec{V} + \vec{W} = (5, 4) + (3, -2)$
$= (5 + 3, 4 - 2)$
$= (8, 2)$

Def(3):- If $U = (U_1, U_2, \ldots, U_n)$ and K is scalar number then

$K U = K(U_1, U_2, \ldots, U_n)$
$= (K U_1, K U_2, \ldots, K U_n)$

Def(4):- If $U = (0, 0, \ldots, 0)$ then U is called zero vector and denoted by $O = (0, 0, \ldots, 0)$, where satisfies $V + O = O + V = V$

ملاحظة:- اذا كان U متجه في R^n فان **النظير الجمعي** للمتجه U هو

$-U = (-U_1, -U_2, \ldots, -U_n)$

والذي يحقق العلاقة

$U + (-U) = O$

<u>Def(5) :-</u> The ***Length***(**Norm**) of the vector $\vec{V}=(v_1, v_2)$ is denoted by $||V||$ and given by

$$||V|| = \sqrt{v_1^2 + v_1^2 + \cdots + v_n^2}$$

Ex:- Find the length of $\vec{U} = (3,4)$

Sol :-

$$||U|| = \sqrt{(3)^2 + (4)^2}$$

$$= \sqrt{9 + 16}$$

$$= \sqrt{25} = 5 .$$

<u>Def(6) :-</u> If $P_1:(v_1,v_2,\ldots,v_n)$, $P_2 : (u_1,u_2,\ldots,u_n)$ are two points in R^n , then the ***Distance*** from P_1 , P_2 is given by $||P_1P_2|| =$

$$\sqrt{(V_1 - U_1)^2 + (V_2 - U_2)^2 + \ldots\ldots + (V_n - U_n)^2)}$$

Ex:- Find the distance from $P_1:(-5,4)$ to $P_2:(-1,2)$

Sol :-

$$||P_1P_2|| = \sqrt{\sqrt{((x_2 - x_1)^2 + (y_2 - y_1)^2)}}$$

$$= (4)^2 + (-2)^2$$

$$= \sqrt{20}$$

Ex:- Find the length of PQ such that

$P:(5,-2)$, $Q:(-1,-7)$

Sol :-

$$||PQ|| = \sqrt{\sqrt{((x_2 - x_1)^2 + (y_2 - y_1)^2)}}$$

$$= \sqrt{((-1 - 5)^2 + (-7 + 2)^2)}$$

$$= \sqrt{(-6)^2 + (-5)^2}$$

$$= \sqrt{36 + 25} = \sqrt{61}$$

Exc :- Find

U -V , 12 V , 4U + 5V , U – 9V and U + V ,where

(1) V = (1 , -9, 0 ,2) و U = (1/4 ,- 2 , 0 , 5)

(2) V = (-2 , 1/3 , 3 , 0 , 1/4) و U = (1/5 , -3 , -1 , 1/3 , 0)

Theorem(4- 1):- If U ,V ,W are vectors in R^n and K , C are scalars numbers then :-

(1) U + V = V + U
(2) (U + V) + W = U + (V + W)
(3) U + O = O + U
(4) U + (-U) = 0
(5) (CK) U = C (KU)
(6) K (U + V) = K U + K V
(7) (C + K) V = C V + K V
(8) 1 . U = U
Proof:- H.W

S_2 -- Scalar Product(Dot product) الضرب العددي

Def:- let U= $(u_1 ,u_2,...,u_n)$ and V =$(v_1 , v_2,...,v_n)$ are two vectors in R^n then the ***Scalar product (Dot product)***defined by

U . V = $u_1 v_1 + u_2 v_2 +\ldots+ u_n+ v_n$

Ex:- Let V=(5,7,1) and U=(-8,0,-12) then find V.U

Sol :-

U . V = (5 (-7) + 7 (0) + 1 (-12))
= -35 + 0 – 12
= -47

Ex:- Let V=(3,-2,5) and U=(-6,1,9) then find the dot product of V and U

Sol :-

U . V = (3 , -2 , 5) . (-6 , 1 , 9)
= -18 -2 + 45
= 25

H.W:- Let V=(7,-9,5) and U=(-20,0,17) then find V.U, V-U ,V+U ,KV+U

The Angle Between Two Vectors الزاوية بين متجهين

Theorem:- if $\vec{U}$ and $\vec{V}$ are non zero vectors in R^2 or R^3 , and if α is the angle between them , then

$$\cos\alpha = \frac{U \cdot V}{\|U\| \cdot \|V\|}$$

Proof

$$\|U - V\|^2 = \|U\|^2 + \|V\|^2 - 2\|U\| \cdot \|V\| \cos\alpha$$

$$\|U - V\|^2 = \left(\sqrt{(U_1 - V_1)^2 + (U_2 - V_2)^2} \right)^2$$

$$= (U_1 - V_1)^2 + (U_2 - V_2)^2$$

$$= U_1^2 + V_1^2 - 2U_1V_1 + U_2^2 + V_2^2 - 2U_2V_2$$

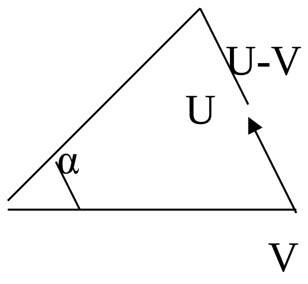
$$= U_1^2 + U_2^2 + V_1^2 + V_2^2 - 2U_1V_1 - 2U_2V_2$$

$$= \|U\|^2 + \|V\|^2 - 2(U_1V_1 + U_2V_2)$$

$$-2\|U\| \cdot \|V\| \cos\alpha = 2U_1V_1 + U_2V_2$$

$$\|U\| \neq 0 \quad , \quad \|V\| \neq 0$$

$$\cos\alpha = \frac{U_1V_1 + U_2V_2}{\|U\| \cdot \|V\|}$$



$$\cos\alpha = \frac{U \cdot V}{\|U\| \cdot \|V\|}$$

Remark:-

من العلاقة اعلاه نستطيع ان نحصل على

$U . V = ||U|| . ||V|| . \cos\alpha$

Ex:- if V=(-3,0) and U =(-3, -4) then find the angle between V ,U

Sol :-

$U . V = -3(-3) + 0(-4)$

$= 9 + 0 = 9$

$||U|| = \sqrt{9 + 16} = \sqrt{25} = 5$

$||V|| = \sqrt{(-3)^2 + (0)^2} = \sqrt{9 + 0} = 3$

$$\cos\alpha = \frac{U . V}{||U|| . ||V||}$$

$$\cos\alpha = \frac{9}{5 . 3}$$

$= 3/5$

$\alpha = \cos^{-1}(3/5)$

Ex:-if V=(0,0,1) and U =(1,0, 0) then find the angle between V ,U

Sol :-

$U . V = (1)0 + 0(0) + (0)1$

$= 0$

$||V|| = \sqrt{0 + 0 + 1^2} = \sqrt{1} = 1$

$||U|| = \sqrt{1^2 + 0 + 0} = \sqrt{1} = 1$

$$\cos\alpha = \frac{U \cdot V}{\| U \| \cdot \| V \|}$$

$= 0 / 1 = 0$

$\cos\alpha = 0 \quad , \quad \alpha = \pi / 2$

TH (4 – 2) :-

If U , V and W are vectors in R^n and K is a scalar , then

a) $\vec{U} \cdot \vec{V} = \vec{V} \cdot \vec{U}$

b) $\vec{U} \cdot (\vec{V} \cdot \vec{W}) = \vec{U} \cdot \vec{V} + \vec{U} \cdot \vec{W}$

c) $K (\vec{U} \cdot \vec{V}) = (K \vec{U}) \cdot \vec{V} + \vec{U} \cdot (K \vec{V})$

d) $V \cdot V = (\| V \|)^2$

e) $V.0 = 0$

TH (4 – 3) :-

V and W are ***orthogonal*** iff $V . W = 0 \quad , V \neq 0 \quad , W \neq 0$

Or if $\cos\alpha = 0 \Longrightarrow \alpha = \pi / 2$

Ex :- let U = (1 , -2 , 2) and V = (2 , 7 , 6) show that U and V are orthogonal vectors .

Sol :-

$U . V = 2 – 14 + 12 = 0$

$$\cos\alpha = \frac{U \cdot V}{\| U \| \cdot \| V \|}$$

$$\cos\alpha = \frac{0}{\| U \| \cdot \| V \|}$$

$\cos\alpha = 0$

$\alpha = \pi / 2$

$U \perp V$

Unit Vector متجـــــه الوحــــــــدة

Def :- If the length of a vector V equal one then V called ***unit vector***

Ex:-
Show that V = (0 , 1) is unit vector?

Sol:
$$\| V \| = \sqrt{0^2 + 1^2} = 1$$

Remark :-

If V is non zero vector then the vector U
$$U = \frac{1}{\| V \|} \cdot \vec{V}$$

Is called unit vector with the ***same direction as V***

Ex:-
Let V = (-5, 7) find the unit vector that has the same direction as V ?

Sol :-

$$\vec{U} = \frac{1}{\| V \|} \cdot \vec{V}$$

$$\| \vec{V} \| = \sqrt{(-5)^2 + (7)^2}$$
$$= \sqrt{25 + 49}$$
$$= \sqrt{74}$$

$$U = 1/\sqrt{74} . (-5 , 7)$$

$$= (-5/\sqrt{74} , 7/\sqrt{74})$$

$$\|\vec{U}\| = \sqrt{(-5/\sqrt{74})^2 + (7/\sqrt{74})^2}$$

$$= \sqrt{25/74 + 49/74}$$

$$= \sqrt{74/74} = \sqrt{1} = 1$$

Ex:-

Let W= (4 , -2 , 1) find the unit vector that has the same direction as W ?

Sol :-

$$U = \frac{1}{\|W\|} \cdot \vec{W}$$

$$\|W\| = \sqrt{(4)^2 + (-2)^2 + (1)^2}$$

$$= \sqrt{16 + 4 + 1} = \sqrt{21}$$

$$U = \frac{1}{\sqrt{21}} \cdot (4, -2, 1)$$

$$= (4/\sqrt{21}, -2/\sqrt{21}, 1/\sqrt{21})$$

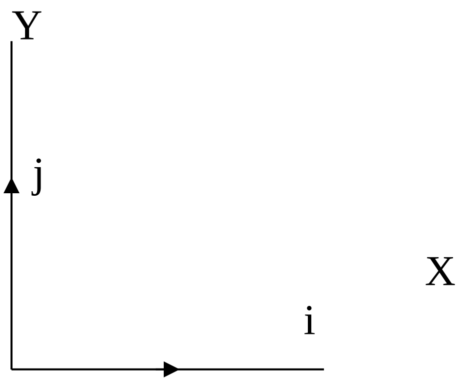
$$\|U\| = \sqrt{(4/\sqrt{21})^2 + (-2/\sqrt{21})^2 + (1/\sqrt{21})^2}$$

$$= \sqrt{16/21 + 4/21 + 1/21}$$

$$= \sqrt{21/21} = 1$$

Remark :-

(1) in R^2 there exist two unit vectors are i = (1 , 0) and j = (0 , 1)



(2) every vector in R^2 is expressible uniquely in terms of i and j as follows

V = (V_1 , V_2) .

= (V_1 , 0) + (0 , V_2)

= V_1 (1 , 0) + V_2 (0 , 1)

= V_1 i + V_2 j

(3) in R^3 there exist three unit vector are

I = (1 , 0 , 0) , j = (0 , 1 , 0) , k = (0 , 0 , 1)

(4) every vector in R^3 is expressible uniquely in terms of i , j , k as follows .

V = (V_1 , V_2 , V_3)

= (V_1 , 0 , 0) + (0 , V_2 , 0) + (0 , 0 , V_3)

= V_1 (1 , 0 , 0) + V_2 (0 , 1 , 0) + V_3 (0 , 0 , 1)

= V_1 i + V_2 j + V_3 k

Ex:-

The vector V= (7 , -9 , 2) we can written as follows .

V = 7 i -9j + 2 k

The Properties Of Unit Vectors

(1)

i . i = 1

j . j = 1

k . k = 1

(2)

i . j = 0

i . k = 0

j . k = 0

Def :- In both R^2 and R^3 the angles between anon zero vector V and the unit vectors i , j , k are called ***direction cosines of V .***

Theorem :- (4 – 5)

The direction cosine of a non zero vector
$V = V_1 I + V_2 j + V_3 k$ are

$$\cos \alpha = \frac{V_1}{||V||}, \quad \cos \beta = \frac{V_2}{||V||}, \quad \cos \theta = \frac{V_3}{||V||}$$

Ex:-
Find the direction cosines of the vector
$V = 2 i - 6 j + 3 k$
Sol :-

$$||V|| = \sqrt{4 + 36 + 9} = \sqrt{49} = 7$$

So that

$$\cos \alpha = \frac{V_1}{||V||} = \frac{2}{7} =$$

$$\cos \beta = \frac{V_2}{||V||} = \frac{-6}{7} =$$

$$\cos \theta = \frac{V_3}{||V||} = \frac{4}{7} =$$

S_3 --Cross Product الضرب الاتجاهى

Def :- if $V = (V_1, V_2, V_3)$, $U = (U_1, U_2, U_3)$ are two vectors in R^3 then the ***Cross Product*** is denoted by $U * V$ and defined as follows :-

$U * V = (U_2 V_3 - U_3 V_2, U_3 V_1 - U_1 V_3, U_1 V_2 - U_2 V_1)$

Or by determinants

$$\vec{U} * \vec{V} = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= i(U_2 V_3 - U_3 V_2) - j(U_1 V_3 - U_3 V_1) + k(U_1 V_2 - U_2 V_1)$$

Ex:-

let V = (1 ,-1 , 0) , U = (2 , 3, -2)

find V*U

Sol :-

$$\vec{U} * \vec{V} = \begin{vmatrix} i & j & k \\ 2 & 3 & -2 \\ 1 & -1 & 0 \end{vmatrix}$$

$= i(2 - 0) - j(-2 - 0) + k(3 + 2)$

$= 2i + 2j + 5k$

TH (4 – 6) :-

If U and V two vectors in R^3 then

a) $\vec{U} * \vec{V} \perp \vec{U}$

b) $\vec{U} * \vec{V} \perp \vec{V}$

c) $\| \vec{U} * \vec{V} \|^2 = \| \vec{U} \|^2 . \| \vec{V} \|^2 - (\vec{U} . \vec{V})^2$

(***Lagranges identity)*** المتطابقة الاخيرة تسمى **متطابقة لاكرانج**

Ex:

If V = (4, 2 , -5) And U = (-2 , 1 , 0)then find

$(\vec{U} * \vec{V}) \perp \vec{U}$

Sol :-

$$\vec{U} * \vec{V} = \begin{vmatrix} i & j & k \\ -2 & 1 & 0 \\ 4 & 2 & -5 \end{vmatrix}$$

$= i(-5) - j(10) + k(-8)$

$= -5i - 10j - 8k$

$\vec{U} \cdot (\vec{U} * \vec{V}) = (-2, 1, 0) \cdot (-5, 10, -8)$

$= 10 - 10 + 0$

$= 0$

TH (4 – 7) :-

If U , V and W are vectors in R^3 and C is a scalar , then

a) $\vec{U} * \vec{V} = -(\vec{V} * \vec{U})$

b) $\vec{U} * (\vec{V} + \vec{W}) = (\vec{U} * \vec{V}) + (\vec{U} * \vec{W})$

c) $(\vec{U} + \vec{V}) * \vec{W} = (\vec{U} * \vec{W}) + (\vec{V} * \vec{W})$

d) $C(\vec{U} * \vec{V}) = (C\vec{U}) * \vec{V} = \vec{U} * (C\vec{V})$

e) $\vec{U} * \vec{O} = \vec{O} * \vec{U} = O$

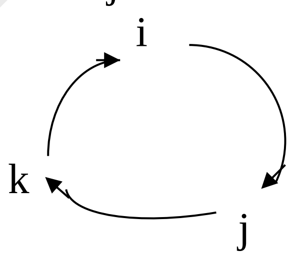
f) $\vec{U} * \vec{U} = 0$

(i , j , k)　　خــواص متجهــــــات الوحـــــــــدة

(1) $i * i = j * j = k * k = 0$

(2) $i * j = k$, $j * k = i$, $k * i = j$

$j * i = -k$, $k * j = -i$, $i * k = -j$



باتجاه الاسهم الاشارة موجبة و عكس اتجاهالاسهم الاشارة سالبة

TH (4 – 7) :-

If U , V be nonzero vectors in R^3 and α be the angle between these vectors then

(a) $\| U * V \| = \| U \| \, \| V \| \sin \alpha$

(b) The ***area*** A of the ***Parallelogram*** that has U and V as adjacent sides is $A = \| U * V \|$

(c) U * V = 0 iff U and V are ***parallel*** vectors

EX;- Find the ***area*** of the ***parallelogram*** determine by the vectors

$\overrightarrow{P_1 P_2} = 3i - j + 2k$

$\overrightarrow{P_1 P_3} = 2i + 3j + k$

Sol :-

$$\overrightarrow{P_1 P_2} * \overrightarrow{P_1 P_3} = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= i(-7) - j(-1) + k(11)$$

$$= -7i + j + 11k$$

$$A = ||U * V||$$

$$= ||\overrightarrow{P_1 P_2} * \overrightarrow{P_1 P_3}||$$

$$||\overrightarrow{P_1 P_2} * \overrightarrow{P_1 P_3}|| = \sqrt{(-7)^2 + (1)^2 + (11)^2}$$

$$= \sqrt{49 + 1 + 121}$$

$$= \sqrt{171}$$

H . W:-

Exc:- (1)

Find the ***area*** of the ***triangle*** that is determined by the points $P_1(1, 2, 0)$, $P_2(-2, 0, 1)$, $P_3(0, 2, 3)$

Exc:- (2)

Find the ***area*** of the ***parallelogram*** determined by the vectors

$P_1P_2 = 3i + 2j - 2k$, $P_1 P_3 = 4i - 5j + 2k$

Exc:- (3)

Find the ***area*** of the ***parallelogram*** that has $U = i - 3j + 2k$ and $V = 3j + 4k$ as adjacent sides .

Exc:- (4)

Find the area of the ***triangle*** with vertices

(a) $P(1, 2, -2)$, $Q(0, 0, 0)$, $R(3, 5, 1)$

(b) $P(2, 0, -3)$, $Q(1, 4, 5)$, $R(7, 2, 9)$

Scalar triple product — الضرب العددي الثلاثي

Def :- let $U = (u_1 , u_2 , u_3)$, $V = (v_1 , v_2 , v_3)$ and $W = (w_1 , w_2 , w_3)$ are vectors in R^3 then the number $U . (V * W)$ is called the ***Scalar triple product*** of U , V and W

Remark :-

The ***Scalar triple product*** can be obtained directly from the formula

$$U . (V * W) = \begin{vmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{vmatrix}$$

Ex :-
Calculate the ***Scalar triple product*** U. (V * W) of the vectors U = 3i – 2j – 5k , V = I + 4 j – 4 k , W = 3 j + 2 k
Sol :-

$$U . (V * W) = \begin{vmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{vmatrix} = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 49$$

Theorm(4-8) :-

Let U , V and W be non zero vectors in R^3
(a) the ***volume*** of the ***parallelepiped*** that has U , V and W as adjacent edges is

$$V = \left| U . (V * W) \right|$$

(b) U . (V * W) = 0 if U , V and W ***lie in the same plane*** .
Ex :-
Let U = (3 , -4 , 1) , V = (0 , 5 , -1) , W = (3 , 0 , -1)
Sol :-

$$V = \left| U . (V * W) \right| = \left| \begin{matrix} 3 & -4 & 1 \\ 0 & 5 & -1 \\ 3 & 0 & -1 \end{matrix} \right| \begin{matrix} 3 & -4 \\ 0 & 5 \\ 3 & 0 \end{matrix}$$

$= \left| -15 + 12 + 0 - 15 \right| = \left| -18 \right| = 18$

V = 16

Ex :-

Let U = (1, -2 , 3) , V = (3 , 2 , -2) , W = (1 , 0 , 3), determine whether the vectors ***lie in the same plane*** .

Sol :-

By Th . the vectors lie in the same plane if U (V * W) = 0

$$V = \left| U . (V * W) \right| = \left| \begin{matrix} 2 & 0 & -3 \\ 3 & 1 & -2 \\ -2 & 0 & 3 \end{matrix} \right| \begin{matrix} 2 & 0 \\ 3 & 1 \\ -2 & 0 \end{matrix}$$

= 6 + 0 + 0 – 6 – 0 - 0 = 0

U . (V * W) = 0 , then these vectors are lie in the same plane .

H. W :-

(1) let U= 3 i + j + 2 k , V = 4 I + 5 j + k , W = 1 + 2 j + 4 k then find the ***volume*** of ***parallelepiped*** that has U , V and W as adjacent edges .

(2) determine whether the vectors U , V and W lie in the ***same plane*** where

(a)U = 5i – 2 j + k , V = 4i – j + k , W = i – j

(b) U= (4 , 8 , 1) , V = (2 , 1 , -2) , W = (3 , -4 , 12)

(3) consider the ***parallelepiped*** with adjacent edges

U= 3i + 8j + k , V = i + j + 2 k , W = i + 3 j + 3 k

(a) find the ***volume***

(b) find the ***area*** of the face determined by U and W

(c) the ***angle*** between U and V

(1) Let U=(1,3) ,V=(2,1) ,W=(4,-1) .

Find the vector **X** that satisfies 2U-V+X=7X+W

(2) Given that K=-2 and ||KV||=6 . Find ||V||

(3) Find the ***initial*** point of the vector V =(-3 ,1,2) , if the ***terminal*** point is (5, 0,-1)

(4) if $||U||=1, ||V|| = 2$,the angle between U and V is (30^0) then find U .V

(5) Use vectors to show that A(2,-1,1) , B(3,2,-1) and C(7,0,-2) are vertices of a ***right triangle*** . ? At which vertex is the ***right angle*** ?

(6) Find K so that the vector from the point A(1,-1,3) to the point B(3,0,5) is ***orthogonal*** to the vector from A to the point (k , k , k)

(7) Find the ***area*** of the ***parallelogram*** that has U and V as adjacent sides

(a) U = i +3j - 2 k and V = 3i-j - k

(b) U =2 i +3 j and V = - i+ 2j - 2 k

(8) Let be the angle between the vectors U=2i+3j-6k and V=2i+3j+6k

(a) use the ***dot*** product to find Cosθ

(b) use the ***cross*** product to find Sinθ

ملاحظه<u>(1)</u>:- جميع الامثله والتمارين والنظريات في الكتاب مطلوبه حول هذا الفصل والفصول الاخرى

<u>S_4 Vector Space فضاء المتجهات</u>

<u>Def;-</u> Let **V** be nonempty set of vectors then **V** called ***Vectors Space*** over R if and only if its satisfy the following conditions

<u>(1)</u>

(a) U , V ∈ V ⟹ U + V ∈ V

V is closed under +

(b) U + V = V + U

(c) U + (V + W) = (U + V) + W

(d) يوجدعنصر وحيد هو المتجه الصفري ينتمي الى V بحيث

U + O = O + U = U

(e) U + (-U) = O لكل U ينتمي الى V فانه يوجد -U ينتمي الى V بحيث انه

(2) if v ,u ∈ V and a , b ∈ R then

(a) a . U ∈

(b) a . (V + U) = a . V + a U

(c) (a + b) . U = a U + b U

(d) a . (b . U) = a b . U

(e) 1 . U = U

Ex:- show thet R^n is vectors space over R ,where
$v+u= (v_1=u_1,v_2+u_2,\ldots,v_n+u_n)$ and $cv =(cv_1,cv_2,\ldots,cv_n)$

Sol:-

By theorem

If U ,V ,W are vectors in R^n and K , C are scalars numbers then :-

(1) U + V = V + U
(2) (U + V) + W = U + (V + W)
(3) U + O = O + U
(4) U + (-U) = 0
(5) (CK) U = C (KU)
(6) K (U + V) = K U + K V
(7) (C + K) V = C V + K V
(8) 1 . U = U

Therefore , R^n is vectors space over R .

Ex:- Let V is the set of all vectors of the form $(U_1,0,U_3)$ and defined the operations of addition and multiplication by scalar number as following

$U+V = (U_1 , O , U_3) + (V_1 , O , V_2) = (U_1 + V_1 , O + O , U_3 + V_3)$

$c.U = C . (U_1 , O , U_2) = (C U_1 , O , C U_2)$. Is V vectors space over R .

Sol :-

لكي نثبت بان V فضاء متجهات يجب ان تحقق لشروط تعريف فضاء المتجهات وكما يلي :-

(1)

(a) $U + V = (U_1 , O , U_3) + (V_1 , O , V_2) = (U_1 + V_1 , O , U_3 + V_3)$

U + V   V

Then V is closed under + .

(b) $U + V = (U_1 , O , U_3) + (V_1 , O , V_3)$

$= (U_1 + V_1 , O , U_3 + V_3)$

$= (V_1 + U_1 , O , V_3 + U_3)$

$= (V_1 , O , V_3) + (U_1 , O , U_3)$

$= V + U$

(c) U + (V + W) = (U + V) + W

Let $U = (U_1 , O , U_3) , V = (V_1 , O , V_3) , W = (W_1 , O , W_3)$

$U + (V + W) = (U_1 , O , U_3) + ((V_1 , O , V_3) + (W_1 , O , W_3))$

$= (U_1 , O , U_2) + (V_1 + W_1 , O , V_3 + W_3)$

$= (U_1 + V_1 + W_1 , O , U_3 + V_3 + W_3)$

$= (U_1 + V_1) + W_1 , O , (U_3 + V_3) + W_3)$

$= ((U_1 + V_1) , O , (U_3 + V_3)) + (W_1 , O , W_3)$

$= ((U_1 , O , U_3) + (V_1 , O , V_3)) + W$

$= (U + V) + W$

(d) U + O = O + U = U

U + O = (U1 , O , U3) + (O, O, O)

= (U1 + O , O + O , U3 + O)

= (U1 , O , U3)
= U

(e) a . (U + V) = a (U1 + V1 , O , U3 + V3)

= (a (U1 + V1) , a . O , a (U3 + V3))

= (a U1 + a V1 , O , a U3 + a V3)

= (aU1 , O , aU3) + (a V1 , O , a V3)

= a U + a V

و هكذا بالنسبة لبقية الشروط
اذن المجموعة V هي فضاء متجهات

<u>Ex:-</u> Let U={ (u1,5,u3,u4) , u1,u3,u4 R } and
U + V = (U1 , U2 , U3,U4) + (V1 , V2 , V3,V4) = (U1 + V1 , U2 + V2 , U3 + V3,U4+V4)
C U = C (U1 , U2 , U3,U4) = (C U1 , C U2 , C U3,CU4)
Is U vectors space

<u>Sol:-</u> H .W

<u>Ex:-</u> If V =M $_{2*3}$(R) ={ set of all matrices of order 2*3 which defined over R} with the addition and multiplication by scalar number of matrices , then show that V is vectors space

Sol :-

Let $U = A = \begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \end{pmatrix}$, $V = B = \begin{pmatrix} b11 & b12 & b13 \\ b21 & b22 & b23 \end{pmatrix}$

(a) $U + V = A + B = \begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \end{pmatrix} + \begin{pmatrix} b11 & b12 & b13 \\ b21 & b22 & b23 \end{pmatrix}$

$$= \begin{pmatrix} a11 + b11 & a12 + b12 & a13 + b13 \\ a21 + b21 & a22 + b22 & a23 + b23 \end{pmatrix}$$

$$= \begin{pmatrix} b11 + a11 & b12 + a12 & b13 + a13 \\ b21 + a21 & b22 + a22 & b23 + a23 \end{pmatrix}$$

$$= \begin{pmatrix} b11 & b12 & b13 \\ b21 & b22 & b23 \end{pmatrix} + \begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \end{pmatrix}$$

= B + A

= V + U

وهكذا بالنسبة لبقية الشروط (برهان بقية الشروط واجب H . W)

اذن المجموعة V هي فضاء متجهات على حقل الاعداد الحقيقية R

Exc:- Determine whether the sets are vectors space

(1) $V = R^2$,with two operations (U1 , U2) + (V1 , V2) = (U1 + V1 , U2 + V2)
k(U.V) = (U.kV)

(2) $M = R^3$,with two operations (x_1, y_1, z_1) + (x_1, y_2, z_2) = ($x_1 + x_2$., $y_1 + y_2$, $z_1 + z_2$)
k(U.V.W) = (0,0,0)

(3) $S = M_{2*2}(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} , a, b \in R \right\}$

with the addition and multiplication by scalar number of matrices.

واجب : - تمارين صفحه (120) وصفحه (123)

(4-1)- SubSpace الفضاء الجزئي

Def:- Let V is vectors space over R ,then the nonempty ***subset U*** of V is called ***subspace*** of V if and only if itself is vectors space

Ex:- Let V is vectors space over R ,then

(1) V is sub space of V

(2) { 0} is sub space of V

Remark :- the two sub space V and{ 0} are called the **trivial subspace** of V Ex:-

Let V = R^4 and W sub set of V such that W={ (0,a,0,b); a,b R }and defined the operations of addition and multiplication by scalar number as following

(0 , a , 0, b) + (0 , x , 0,y) = (0+0, a + x , O + O , b+ y)

C . (0 , a , 0,b) = (0 , c.a , 0,cb). Is W subspace of V ?

Sol:-

يجب ان نطبق شروط تعريف الفضاء على W .

(1) Let U ϵ W ⟶ U= $(0,u_2 , o ,u_4)$

V ϵ W ⟶ V = $(0,v_2 , o ,v_4)$

U + V = $(0,u_2 , o, u_4)+ (0,v_2, o,o ,v_4) = (0+0,u_2+v_2 ,o,u_4+v_4)$ ϵ W

وبقية الشروط نفس الحالة .

therefore W is subspace of R^4

اذن W يمثل فضاء جزئي من R^4 .

Theorem (4-9) :- Let W be sub set of the vectors space V then W is sub space of V iff

(1) W is closed under the addition ;(if u ,v ϵ W ,then u + v ϵ W)

(2) W is closed under the multiplication by scalar number; (if u ϵ W and c ϵ R ,then cu ϵ W) .

Ex:- Let W = $\left\{ \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \\ a_{31} & 0 \end{pmatrix} ; a_{11}, a_{22} , a_{31} \in R \right\}$

and V = M_{3*2} (R) , then show that W is subspace of V

Sol :-

الحل :-

لكي نثبت ان W فضاء جزئي يجب ان نحقق شرطي النظرية السابقة .

Let

A = $\begin{pmatrix} a_{11} & o \\ o & a_{22} \\ a_{13} & 0 \end{pmatrix}$, B = $\begin{pmatrix} b_{11} & o \\ o & b_{22} \\ b_{13} & 0 \end{pmatrix}$

(1)

A + B = $\begin{pmatrix} a_{11} & o \\ o & a_{22} \\ a_{13} & 0 \end{pmatrix}$ + $\begin{pmatrix} b_{11} & o \\ o & b_{22} \\ b_{13} & 0 \end{pmatrix}$

$$= \begin{pmatrix} a_{11} + b_{11} & o \\ o & a_{22} + b_{22} \\ a_{31} + b_{31} & 0 \end{pmatrix}$$

هذا الشكل ينتمي الى W

اذن A + B ينتمي الى W

(2)

$$C\,A = C \cdot \begin{pmatrix} a_{11} & o \\ o & a_{22} \\ a_{31} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} c\,a_{11} & o \\ o & c\,a_{22} \\ c\,a_{31} & 0 \end{pmatrix}$$

هذا الشكل ينتمي الى W

اذن C A ينتمي الى W

اذن W فضاء جزئي من V

Ex:-

Let $V = R^4$ and W sub set of V such that $W=\{ (a,b,7,d);\ a,b,d \in R\}$ with the operations of addition and multiplication by scalar number

Is $(W ,+ ,.)$ subspace of R^4 ?

Sol :- H.W

واجب :- تمارين صفحه (132 -133) التمرين الاول والثاني

TH(4-10) ;- If U , W are two sub space of V then U + W is subspace of V

Ex:-

Let $V = M_{2*2}(R)$ and

$$U = \left\{ \begin{pmatrix} o & y \\ o & n \end{pmatrix} ,\quad y, n \in R \right\}$$

$$W = \left\{ \begin{pmatrix} o & o \\ x & m \end{pmatrix} ,\quad x, m \in R \right\}$$

$$U + W = \left\{ \begin{pmatrix} o & y \\ x & z \end{pmatrix} ,\quad x, y, z \in R ,\ z = m + n \right\}$$

Then show that U+ W is subspace of V ?

Sol :- **الحل :-**

لكي نثبت ذلك يجب ان نحقق شروط النظرية السابقة

Let $u_1, u_2 \in U$, $w_1, w_2 \in W$

$\longrightarrow u_1+w_1 \in U+W$, $u_2+w_2 \in U+W$

To prove that

$(u_1+w_1) + (u_2+w_2) \in U+W$ الان يجب ان نبرهن بان

(1)

$$(u_1 + w_1) + (u_2 + w_2) = \left\{\begin{pmatrix} o & y_1 \\ o & n_1 \end{pmatrix} + \begin{pmatrix} o & o \\ x_1 & m_1 \end{pmatrix}\right\} + \left\{\begin{pmatrix} o & y_2 \\ o & n_2 \end{pmatrix} + \begin{pmatrix} o & o \\ x_2 & m_2 \end{pmatrix}\right\}$$

$$= \begin{pmatrix} o & y_1 \\ x_1 & n_1+m_1 \end{pmatrix} + \begin{pmatrix} o & y_2 \\ x_2 & n_2+m_2 \end{pmatrix}$$

$$= \begin{pmatrix} o & y_1 \\ x_1 & z_1 \end{pmatrix} + \begin{pmatrix} o & y_2 \\ x_2 & z_2 \end{pmatrix}$$

$$= \begin{pmatrix} o & y_1+y_2 \\ x_1+x_2 & z_1+z_2 \end{pmatrix}$$

$$= \begin{pmatrix} o & y \\ x & z \end{pmatrix}$$

اذن $\begin{pmatrix} o & y \\ x & z \end{pmatrix}$ ينتمي الى $U + W$

(2)

Let $u_1 = \begin{pmatrix} o & y_1 \\ o & n_1 \end{pmatrix}$, $w_1 = \begin{pmatrix} o & o \\ x_1 & m_1 \end{pmatrix}$

$$C (u_1 + w_1) = C \left(\begin{pmatrix} o & y_1 \\ o & n_1 \end{pmatrix} + \begin{pmatrix} o & o \\ x_1 & m_1 \end{pmatrix} \right)$$

$$= C \begin{pmatrix} o & y_1 \\ x_1 & n_1 + m_1 \end{pmatrix}$$

$$= C \begin{pmatrix} o & y_1 \\ x_1 & z_1 \end{pmatrix}$$

$$= \begin{pmatrix} o & c\,y_1 \\ c\,x_1 & c\,z_1 \end{pmatrix} \in U + W.$$

Therefore $U + W$ is subspace of V .

<u>Theorem(4-11)</u> :- If U , W are two subspace of V then U∩ W is subspace of V

<u>Proof :-</u> since

$0 \in U$, $0 \in W \longrightarrow 0 \in W \cap U$

then , $W \cap U$ is nonempty set .

(1) Let

$u,v \in W \cap U$

$\longrightarrow u,v \in W$, $u,v \in U$

$\longrightarrow u+v \in W$, $u+v \in U$

$\longrightarrow u+v \in W \cap U$.

(2))

Let $u \in W \cap U$ and $k \in R$.

Then, because W and U are subspace ,we have $ku \in W$, and $ku \in U$.

Thus, $ku \in W \cap U$

Therefore , by (1) and (2), we get $W \cap U$ is subspace of V

(4-2)Direct Sum الجمع المبا شر

Def:- let V be a vectors space and U ,W are two sub space of V ,then we say that V is ***Direct Sum*** of U and W if and only if **every vectors in V can be written by sum of two vectors the first from U and the second from W such that the representation is uniquely** ,and denoted by **$V = U \oplus W$**

Ex:-

Let $V = R^3$ and U , W are two sub space of V such that

$U = \{ (a, b, o) , a, b \in R \}$

$W = \{ (o, b, c) , b, c \in R \}$

Determine whether V is ***direct sum*** of U and W ?

Sol :-

Let $(x, y, z) \in R^3$

$(x, y, z) = (x, y, o) + (o, o, z)$

is not direct sum

since $(4, -6, 3) = (4, -6, 0) + (0, 0, 3)$

$= (4, -3, 0) + (0, -3, 3)$

$= (4, -1, 0) + (0, -5, 3)$

the representation is not uniquely اذن التمثيل ليس وحيدا

$R^3 \neq U \oplus W$

Ex:-

Let $V = R^4$ and U , W are two sub space of V such that

$U = \{(a, 0, 0, d) \quad , a, b, d \in R\}$

$W = \{(0, b, c, 0) \quad , b, c \in R\}$

Determine whether V is ***direct sum*** of U and W ?

Sol :-

Let $(x, y, z, r) \in R^4$

$(x, y, z, r) = (x, 0, 0, r) + (o, y, z, 0)$

since , the representation is uniquely

then, V is direct sum of U and W

<u>TH(4-12)</u> ;- If U , W are two subspace of V then V is direct sum of U and W ($V = U \oplus W$) iff

(1) $V = U + W$

(2) $U \cap W = \{0\}$

Ex:-

Let $V = R^5$ and U , W are two sub space of V such that

$U = \{(a, 0, c, 0, e) \quad , a, c, d, e \in R\}$

$W = \{(0, b, c, d, 0) \quad , b, c, e \in R\}$

Determine whether V is ***direct sum*** of U and W ?

Sol :-

Let $(x, y, z, r, m) \in R^5$

By th (4-12)

(1) $(x, y, z, r, m) = (x, 0, z, 0, m) + (o, y, 0, r, 0)$

(2) $U \cap W = \{c\} \neq \{0\}$

then, V is not direct sum of U and W

(4-4) Linear Combination التركيب الخطى

-

<u>Def:-</u> Let V be a vectors space over R .Then we say $v \in V$ is ***Linear Combination*** of the vectors $v_1, v_2, \ldots, v_n$ (where $v_i \in V$, for each $i=1,\ldots,n$) if can be written as following $v = k_1 v_1 + k_2 v_2 + \ldots k_n v_n$,where $k_1, k_2, \ldots k_n$ are scalar numbers.

<u>Ex:-</u> Let $V_1 = (1,2,1,-1)$,$V_2 = (1,0,2,-3)$,$V_3 = (1,1,0,-2)$ are vectors in R^4 show that the vector $V = (2,1,5,-5)$ is linear combination of V_1, V_2, V_3 ?

Sol :-

By Def حسب التعريف اعلاه

$V = c_1 V_1 + c_2 V_2 + c_3 V_3$

To find c_1, c_2, c_3

$(2, 1, 5 , -5) = c_1 (1 , 2 , 1, -1) + c_2 (1 , 0 , 2 , -3) + c_3 (1, 1 , 0 , -2)$

$= (c_1 , 2 c_1, c_1, -c_1) + (c_2 , 0 , 2c_2 , -3c_2) + (c_3, c_3 , 0 , -2c_3)$

$(2 , 1, 5 , -5) = (C_1 + C_2 + C_3 , 2C_1 + C_3 ,C_1+ 2C_2 , -C_1 -3 C_2 -2 C_3)$

$C_1 + C_2 + C_3 = 2$ ……………..(1)

$2C_1 + C_3 = 1$ …………….(2)

$C_1 +2C_2 = 5$ ……………….(3)

$-C_1 - 3C_2 -2C_3 = -5$ ………………..(4)

تحل هذه المعادلات باستخدام الطرق المناسبة السابقة .

بعد حلها بطريقة كاوس نحصل على النتائج التالية $C_3 = -1 , C_2 = 2 , C_1 = 1$

اذن المتجه V هو تركيب خطي من المتجهات V_1 , V_2 , V_3

اي ان $V= 1 . V_1 + 3 V_2 - 2 V_3$

<u>Ex:-</u> Let $U_1= (1,2,-1)$,$U_2=(1,0,1)$ are vectors in R^3 . Determine whether $U = (1,0,2)$ is linear combination of U_1,U_2?

Sol :-

Let a ,b R then

$U = a U_1 + b U_2$

$U = a (1 , 2 , -1) + b (1 , 0 ,1)$

$(1 , 0 ,2) = (a + b , 2a , - a + b)$

therefore اذن

$a + b = 1$ ……………(1)

$2 a = 0$ ………………….(2)

$- a + b = 2$ …………………..(3)

بعد حل المعادلات نحصل على $a = 0 , b = 1$ وكذلك $b = 2$ وهذا غير ممكن اذن لايوجد حل لهذا النظام .

U is not linear combination of U_1,U_2

<u>Ex:-</u> Let $V_1= (1,-2,0,3)$,$V_2=(2,3,0 ,-1)$,$V_3 =(2,-1,2,1)$ are vectors in R^4 show that the vector $V = (3,9,-4,-2)$ is linear combination of V_1,V_2, V_3 ?

H..W-

واجب :- تمارين صفحه (132 -133) التمرين الثالث والرابع

(4-5) Span – Generate of vector spase   مولد فضاء المتجهات V

Def :- Let $S = \{ v_1, v_2, \ldots, v_n \}$ be a subset of a vectors space V then we sat that S is ***Generate (Span)*** V if **every vector of V is a linear combination of $S = \{ v_1, v_2, \ldots, v_n \}$**

Ex:- Let $V = R^3$ and $S = \{v_1, v_2, v_3\}$ such that $V_1 = (1, 2, 1)$, $V_2 = (1, 0, 2)$, $V_3 = (1, 1, 0)$. Does S generate V ?

Sol :-

لكي نثبت بان S تولدV يجب ان نثبت ان كل متجه ينتمي الى V هو تركيب خطي من عناصر S وكما يلي

Let $v \in V \longrightarrow v = (a, b, c)$

By Def of linear combination

$v = k_1 v_1 + k_2 v_2 + k_3 v_3$

$(a, b, c) = k_1 (1, 2, 1) + k_2 (1, 0, 2) + k_3 (1, 1, 0)$

وبعد حل هذه المعادلة كما حصل في طريقة التركيب الخطي نحصل على المعادلات الاتية

$K_1 + K_2 + K_3 = a$

$2K_1 + K_3 = b$

$K_1 + 2K_2 = c$

الان ناخذ مصفوفة المعاملات

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

ثم نجد لها المحدد

(أ) اذا كان محددها يساوي صفر فانها غير قابلة للانعكاس وبالتالي ليس لها معكوس ومن نظرية سابقة ليس لهذا النظام حل ومنه نحصل على ان S لاتولد V وينتهي الحل .

(ب) اما اذا كان المحدد لايساوي صفر فان A قابلة للانعكاس اي يوجد معكوس ومنه نحصل على ان هذه المعادلات لها حل وبالتالي سوف نحصل على ان S تولد V وينتهي الحل .

الان نجد المحدد للمصفوفة A بالطرق السابقة

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix} \begin{matrix} 1 & 1 \\ 2 & 0 \\ 1 & 2 \end{matrix}$$

$= 0 + 1 + 4 - 0 - 2 - 0 = 3 \neq 0$

$|A| \neq 0$

Thus, A^{-1} is exists ,and hence, there exists solution of this system.

hence , **every vector of V is a linear combination of $S = \{ v_1, v_2, v_3 \}$**

therefore , S is **generated** of R^3.

Ex:- Let $V = R^2$ and $S = \{i, j\}$,show that S is generated R^2

Sol:-

Let $V \in R^2 \longrightarrow V = (a, b)$

$V = K_1 V_1 + K_2 V_2$

$(a, b) = K_1 (1, 0) + K_2 (0, 1)$

$(a, b) = (K_1, 0) + (0, K_2)$

$(a, b) = (K_1, K_2)$

$\longrightarrow a = K_1 \longrightarrow K_1 = a$

$\longrightarrow b = K_2 \longrightarrow K_2 = b$

Then there exists solution to this system, and hence , **every vector of V is a linear combination of $S = \{v_1, v_2\}$.**
Therefore , S is **generated** of R^2

Ex:- Let $V = R^2$ and $S = \{i, j, k\}$,show that S is generated R^3

Ex:- Let $V = R^3$ and $S = \{v_1, v_2, v_3\}$ such that $V_1 = (3, 1, 2)$, $V_2 = (1, 0, 1)$, $V_3 = (2, 5, 3)$. Does S generate V ?

ملاحظة :- **ان متجهات الوحدة لكل فضاء تولد عناصر ذلك الفضاء**

(4-6) Linearly Independent & Linearly Dependent — الاستقلال الخطي والارتباط الخطي

Def :- Let $S = \{v_1, v_2, \ldots\ldots, v_n\}$ be a subset of a vectors space V then we say that S is

(1) **Linearly Dependent** if there exist elements $k_1, k_2, \ldots\ldots, k_n$ in R such that **not all equal to zero with $k_1 v_1 + k_2 v_2 + \ldots\ldots + k_n v_n = 0$**

(2) **Linearly Independent** if there exist elements $k_1, k_2, \ldots\ldots, k_n$ in R such that **all equal to zero with $k_1 v_1 + k_2 v_2 + \ldots\ldots + k_n v_n = 0$**

Ex:- Let $S = \{v_1, v_2, v_3\}$ such that $V_1 = (1, 0, 2)$, $V_2 = (0, -1, 3)$, $V_3 = (-2, 0, 1)$ are vectors in R^3 . Determine whether S is Linearly Independent or Linearly Dependent ?

Sol :-

لكي تكون S مستقلة خطيا يجب انه تطبق المعادلة

$K_1 V_1 + K_2 V_2 + K_3 V_3 = 0$

ونحصل منها على ان جميع الاعداد K1 , K2 , K3 تساوي اصفار وعكس ذلك سوف تكون مرتبطة خطيا

$K_1 V_1 + K_2 V_2 + K_3 V_3 = 0$

$K_1 (1, 0, 2) + K_2 (0, -1, 3) + K_3 (-2, 0, 1) = (0, 0, 0)$

ومن حل هذه المعادلات نحصل على المعادلات الخطية التالية

$K_1 - 2K_3 = 0$ ....(1)

$-K_2 = 0$ ....(2)

$2K_1 + 3K_2 + K_3 = 0$ ....(3)

وبحل المعادلات اعلاه بالطريقة السابقة نحصل على

$K_1 = K_2 = K_3 = 0$

Therefore , $S = \{ V_1, V_2, V_3 \}$ is linearly independent

Ex:- Does the vectors $V_1 = (1, -1)$, $V_2 = (2, -3)$, $V_3 = (5, 1)$ are linearly dependent

Sol:- **الحل :-** نطبق المعادلة التالية

$K_1 V_1 + K_2 V_2 + K_3 V_3 = 0$

ولكي تكون V_1, V_2, V_3 مرتبطة خطيا يجب ان تكون الاعداد K_1, K_2, K_3 على الاقل واحد منها لايساوي صفر

$K_1 V_1 + K_2 V_2 + K_3 V_3 = 0$

$K_1 (1, -1) + K_2 (2, -3) + K_3 (5, 1) = (0, 0)$

من حل هذه المعادلة نحصل على نظام المعادلات الخطية التالي

$K_1 + 2K_2 + 5K_3 = 0$ ....(1)

$-K_1 - 3K_2 + K_3 = 0$ ....(2)

هذا النظام مكون من معادلتين وثلاث متغيرات فيكون له ما لا نهاية من الحلول ولايجاد احد هذه الحلول نفرض ان K_1 يساوي قيمة اختيارية ثم نجد بدلالتها K_2, K_3 وكما يلي

Let $K_3 = 1$

بالتعويض بالمعادلات اعلاه نحصل على

$K_1 = -17$, $K_2 = 6$

وبهذا نستنتج بان V_1, V_2, V_3 مرتبطة خطيا .

ملاحظة :- تحقق من صحة الحل

لانه لو عوضنا عن قيمة (K_1, K_2, K_3) (ليست جميعها اصفار) نحصل على

$(-17)(1, -1) + (6)(2, -3) + (1)(5, 1) = (-17, 17) + (12, -18) + (5, 1) = (0,0)$

وينتهي الحل

Exc:- (1) Determine whether the following sets S is **Linearly Independent** or **Linearly Dependent** ?

(a) S = (6 , 2 , 3 , 4) , (0 , 5 , -3 , 1) , (0 , 0 , 7 , -2)

(b) S = (1, -1 , 0) (1 , 3 , -1) , (5 , 3 , -2)

(c) S = (1, 1, 1, 1) , (2, 3, 1, 2) , (3, 1 , 2 ,1) , (2, 2, 1, 1)

(d)S = (1, 0, 0) , (0 , 1, 0) , (0, 0, 1) , (2, 3, -5)

(e) S = (1, 2, 5) , (1 , 2, -1)

(2) Show that S= { i,j, k } is **Linearly Independent**

(3) Determine whether the matrix B is **linear combination** of the matrices A1 ,A2 ,A3 such that

$$A1 = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}, \quad A2 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \quad A3 = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$$

Where

(a) $B = \begin{pmatrix} 3 & -1 \\ 3 & 2 \end{pmatrix}$

(b) $B = \begin{pmatrix} 5 & 1 \\ -1 & 9 \end{pmatrix}$

(c) $B = \begin{pmatrix} 3 & -2 \\ 3 & 2 \end{pmatrix}$

(4) Determine whether the following sets S is **generate** of R^3

(a) S = (3 , 0 , 3) , (2 , 2 , 0) , (1 , 1 , 1)

(b) S = (2 , -1 , 3) , (4 , 1 , 2) , (8 , -1 , 8)

(4) Determine whether the following sets S is **generate** of R^4

(a) S = ((1, 0, 0, 1) , (0, 1, 0, 0) , (1 , 1 , 1 , 0) , (1 , 1 , 1 , 1)

(b) S = (2 , 0 , 0 , 1) , (6 , 4 , -2 , 4) , (5 , 6 , -3 , 2) , (3 , 2 , -1 , 2) , (0 , 4 , -2 , -1)

(c) S = (1 , 1 , -1 , 0) , (1 , 2 , 1 , 0) , (0 , 0 , 0 , 1)

(4-7)Basis & Dimension الاساس والبعد

Def :- Let S ={ v_1 ,v_2 ,…….,v_n } be a subset of a vectors space V then S is called ***Basis*** for V if

(1) S is ***spans*** (***generate***) V

(2) S is ***Linearly Independent***

Ex:- Let V= R^2 and S={i ,j } ,show that S is Basis for R^2

الحل :- يجب ان نطبق شرطي التعريف اعلاه

(1)To prove that S Linear Ind.

K_1 i + K_2 j = 0

K_1 (1 , 0) + K_2 (0 , 1) = (0 , 0)

(K_1 , 0) + (0 , K_2) = (0 , 0)

(K_1 K_2) = (0 , 0)

K_1 = 0 & K_2 = 0

Therefore S is Linearly Independent

(2)To prove that Sis span set.

Let $W \in R^2$ then $W = (w_1 , w_2)$

Now, let $ai + bj = w$

$a (1 , 0) + b (0 , 1) = (w_1 , w_2)$

$(a , 0) + (0 , b) = (w_1 , w_2)$

$(a , b) = (w_1 , w_2)$

$a = w_1$ ……………………(1)

$b = w_2$ ……………………(2)

اذن هذا النظام له حل اذن كل متجه في R^2 هو تركيب خطي من عناصر S وبالتالي S تولد R^2

By (1) , (2) then S is Basis for R^2

ملاحظه :-

ويطلق على { i ,j } القاعدة الاعتيادية (Standard Basis) في R^2

Ex:- Let $V= R^4$ and $S=\{v_1 ,v_2 ,v_3,v_4 \}$ such that $V_1 = (1 , 0 , 1,0)$, $V_2 = (0 , 1, -1,2)$, $V_3 = (0 , 2 , 2,1)$ $v_4 = (1 ,0,0,1)$. Does S Spans R^4 ?

Sol :-

(1) To show that S Linear Ind.

Let $K_1 V_1 + K_2 V_2 + K_3 V_3 + K_4 V_4 = 0$

نعوض عن قيم V_1 , V_2 , V_3 , V_4 كما مر سابقا فنحصل على نظام المعادلات الخطية التالية

$K_1 + K_4 = 0$

$K_2 + 2 K_3 = 0$

$K_1 – K_2 + 2 K_3 = 0$

$2 K_2 + K_3 + K_4 = 0$

بما ان النظام نظاما متجانسا وعدد المعادلات يساوي عدد المتغيرات فانها تمتلك حل وحيد و هو الحل الصفري اي ان

$K_1 = K_2 = K_3 = K_4 = 0$

Therefore , S is linearly Ind.

(1) To show that S Spans for R^4

Let $V = (a , b , c , d)$ R^4

$V = K_1 V_1 + K_2 V_2 + K_3 V_3 + K_4 V_4$

$(a , b , c , d) = K_1 (1 , 0 , 1 , 0) + K_2 (0 , 1 , -1 , 2) + K_3 (0 , 2 , 2 , 1)+ K_4 (1 , 0 , 0 , 1)$

وبحل هذه المعادلة نحصل على نظام المعادلات الخطية التالية :-

$K_1+ K_4 = a$ ………….(1)

$K_2 + 2 K_3 = b$ ………(2)

$K_1 – K_2 + 2 K_3 = c$ …….(3)

$2 K_2 + K_3 + K_4 = d$ …….(4)

ولحل هذا النظام يجب ان ناخذ مصفوفة المعاملات حسب نظرية سابقة (فان لهذا النظام حلا اذا وفقط اذا كانت مصفوفة المعاملات قابلة للانعكاس وفي نظرية اخرى تكون مصفوفة المعاملات قابلة للانعكاس اذا كان محددها لا يساوي صف)

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

نجد المحدد بالطرق السابقة

$|A| \neq 0$

$\longrightarrow$ A^{-1} there exists ,then there exists solution to this system

hence , **every vector of V is a linear combination of S= { v1 ,v2 v3 }**

therefore , S is **generated** (Spans) R^4

by (1) , (2) then S is Basis for R^4,

<u>Exc:-</u> (1) Show that S is basis for $V = M_{2*2}(R)$ such that

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

(2) Show that S= { i ,j, k } is basis for R^3

(3) Determine whether the following sets S is basis for R^3

(a) S= { (0,1,-1), (4,1,-1), (2,3,4), (1,1,-1)}

(b) S={(0,1,0), (-1,2,1) ,(3,2,2)}

(c) S={1,6,4),(2,4,-1),(-1,2,5)}

(4) Determine whether the following set S is basis for $V = M_{2*2}(R)$ such that

$$S = \left\{ \begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -8 \\ -12 & -4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \right\}$$

(3) Determine whether the following sets U is sub space of the vectors space R^3

(A) U= { (x, y ,z) , such that x + y +z = 0 }

(b) U= { (x, y ,z) , such that x = y and 2y = z }

(c) U = { (x, y ,z) , such that x + y =3z }

<u>Def :-</u> Let V be a vectors space over R ,then **the number of its basis** is called ***the Dimension of V*** and denoted by **dim(V) .**

<u>Ex-:-</u> Let V = { 0} , Find ***the Dimension of V?***

Sol :- since V ={0} is L . D then

The basis of V = ∅

So that dim(V) = 0

<u>Ex-:-</u> Let V = R^2 , Find ***the Dimension of V?***

<u>Sol</u> : The basis is

B= { (1,0), (0,1) } = { i,j }

So that dim(V) = dim (R^2) =2

<u>Ex-:-</u> Let V = R^3 , Find ***the Dimension of V?***

B= { (1,0,0) ,(0,1,0) , (0,0,1) } = { i,j, k }

Dim(R^3) = 3

<u>Remark:-</u> in general dim (R^n) = n.

<u>Theorem (4-13):-</u> Let U be a sub space of a vectors space V such that dim(V)=n ,then dim(U)< n and if V =U then dim(U)= n

<u>Ex:-</u> Let U be a sub space of R^3 ,Then find dim(U)?

<u>Sol:-</u> since dim(R^4)= 4 and by above theorem ,then

Dim U= (0) , or (1) or (2) or (3)or (4)

a) dim U = 0 ⟶ U =(0)

b) dim U = 1 ⟶ U= (a,0,0,0)

c) dim U = 2 ⟶ U= (a,b,0,0)

d) dim U= 3 ⟶ U= (a,b,c,0)

d) dim U= 4 ⟶ U= (a,b,c,d) = R^4

<u>Theorem (4-14):-</u> Let U and W be a sub space of a vectors space V and for each of them finite dimension then

Dim(U+W)= dim (U) + dim (W) – dim (U ∩ W)

<u>Remark:-</u> If V= U ⊕ W then Dim (V) = dim (U)+ dim (W)

<u>Ex:-</u>

Let V=R^3 and U ,W two sub space of V such that

U={(a,b,0), a,b ϵ R } = xy plane

W= { (0,b,c), b,c, ϵ R} = yz plane

Then find dim(U + W) ?

Sol:

Dim (U) =2

Dim (W) =2

Dim (U ∩ W) = 1

Dim (U +W) = dim (U) + dim (W) – dim (U ∩ W)

= 2=2-1

= 3

Ex:- (1) let S= { i , j } is a basis of R^2

ϵand let W = { (x , y) R^2 , x + y = 0 }

(a) show that W is sub space of R^2

(b) find the dim (W)

Sol :- H.W

<u>Remark :-</u>

Let S = { v_1 , v_2 , ……….., vn } is basis of vector space of V and if V is any vector in V then we can written as linear combination of the vectors of S as following

$V = k_1v_1 + k_2 v_2 + \ldots\ldots + kn\ vn$, where k_1 , k_2 , …….., kn are scalar number , then we shall called the vector (k_1 , k_2 , ……. , kn) by the ***coordinates vector of V*** and denoted by $\{ V \}$ ***s***

<u>Ex:-</u> let V = R^3 , find the ***coordinates vector*** of the vector V = (3 , 1 , -4) with the basis S = { v_1 , v_2 , v_3 } such that

V_1 = (1 , 1 , 1) , V_2 = (0 , 1 , 1) , V_3 = (0 , 0 , 1)

Sol :-

لتكن V كتركيب خطي من V_1 , V_2 , V_3 اي انه يوجد اعداد k_1 , k_2 , k_3 بحيث ان

$(3 , 1 , -4) = k_1 (1 , 1 , 1) + k_2 (0 , 1 , 1) + k_3 (0 , 0 , 1)$

تحل هذه المعادلة كما مر سابقا ثم نحصل على المعادلات التالية

$K_1 = 3$

$K_1 + k_2 = 1$

$K_1 + k_2 + k_3 = -4$

ثم تحل هذه المعادلات ونحصل منها على

$k_1 = 3$, $k_2 = 2$, $k_3 = -5$

Then the coordinates vector is $\{ V \}$ s = (3 , 2 , -5)

Chapter Five

Inner Product Spaces فضاءات الجداء الداخلي

Def :- let V vector space then the ***inner product*** on V is function from V* V to the filed R and denoted by < u , v > , where u , v ϵ V , then

(1) In R^2 , we have < u , v > = $u_1 v_1 + u_2 v_2$

(2) In R^3 , we have < u , v > = $u_1 v_1 + u_2 v_2 + u_3 v_3$

(3) In R^n , we have < u , v > = $u_1 v_1 + u_2 v_2 + \ldots\ldots + u_n v_n$

Theorme :- (Cauchy – Schwarz Inequality)

Let U , V any two vectors in inner product space V then

$< U , V >^2 \; < \; < U , U > . < V , V >$

Ex :-

Let U = (2 , 1 , -4) , V = (-1 , 2 , 1) be a vectors in R^3 show that U , V is satisfy (Cauchy – Schwarz Inequality)

Sol :- (Cauchy – Schwarz Inequality)

$< U , V >^2 \; < \; < U , U > . < V , V >$

now < U , V > = (2) (-1) + (1) (2) + (-4) (1) = -4

< V , V > = (- 1) (- 1) + (2) (2) + (1) (1) = 6

< U , U > = (2) (2) + (1) (1) + (-4) (-4) = 21

$< U , V >^2 \; < \; < U , U > . < V , V >$

$(-4)^2 < 6 (21)$

16 < 126

Def :- let V be vector space . then we say that S is ***orthonormal*** set if

(1) S is ***orthogonal*** set

(2) the ***length*** of any vector in S ***equal one*** .

Ex:- let S= (U_1 , U_2 , U_3) such that

U_1 = (1 / 3 , 1 / 3 , 1 / 3) , U_2 = (-2 / 6 , 1 / 6 , 1 / 6) , U_3 = (0 , 1 / 2 , -1 / 2)

show that S is orthonormal set

Sol :-(a)

(1) < U1 , U2 > = (-2 / 18 + 1 / 18 + 1 / 1 8) = 0

(2) < U1 , U3 > = 0 + 1 / 6 - 1 / 6 = 0

(3) < U2 , U3 > = 0 + 1 / 12 - 1 / 12 = 0

S is orthogonal set

(b)

$$||U_1|| = \sqrt{< U_1 , U_2 >} = \sqrt{1/3 + 1/3 + 1/3} = \sqrt{3/3} = 1$$

$$||U_2|| = 1$$

$$||U_3|| = 1$$

By (a) and (b) then S is orthonormal set

Ex :- let S = ((1 , 1 , 1) , (0 , 0 , 0)) determine whether S is orthonormal set

Sol :- H . W

Gram – Schmidt Process خطوات كرام - شميدت

ليكن V فضاء جداء داخلي غير خال بعده (n) وان $S= \{ U_1 , U_2 , ... , U_n \}$ اي قاعدة في V لكي نحصل على قاعدة معيارية (orthonormal basis)

$S^* = \{ V_1 , V_2 , ... , V_n \}$

نتبع خطوات كرام شميدت التالية

الخطوة الاولى :-

(1) to find (V_1)

نفرض ان

Let

$$V_1 = \frac{U_1}{||U||}$$

$||V_1|| = 1$ (V_1 متجه طوله واحد)

(2) to find (V_2) **الخطوة الثانية :-**

للحصول على V_2 بحيث يكون طوله واحد وعمودي على V1 يتم ذلك كما يلي (تحسب مركبة U_2 العمودية على الفضاء W_1 المتولد في V_1 ثم نجعل معياره (طوله يساوي واحد) اي ان

$$V_2 = \frac{U_2 - < U_2 , V_1 > V_1}{|| U_2 - < U_2 , V_1 > V_1 ||}$$

(3) to find (V_3) **الخطوة الثالثة :-**

للحصول على V_3 بحيث يكون طوله واحد وعمودي على كل من V_1 , V_2 يتم ذلك كما يلي

(تحسب مركبة المتجه U_3 العمودية على الفضاء W_2 المتولد من V_1 , V_2 ثم نجعل معياره واحد) اي ان

$$V_3 = \frac{U_3 - < U_3 , V_1 > V_1 - < U_3 \; V_2 > V_2}{|| U_3 - < U_3 , V_1 > V_1 - < U_3 , V_2 > V_2 ||}$$

الخطوة الرابعة :- **(4) to find (V_4)**

للحصول على المتجه V_4 والذي معياره (طوله) واحد والعمودي على كل من V_1 , V_2 , V_3 يتم ذلك كما يلي (تحسب مركبة U_4 العمودية على الفضاء W_3 المتولد من V_1 , V_2 , V_3 ونجعل معياره واحدا) اي ان

$$V_4 = \frac{U_4 - \langle U_4 , V_1 \rangle V_1 - \langle U_4 , V_2 \rangle V_2 - \langle U_4 , V_3 \rangle V_3}{\left\| U_4 - \langle U_4 , V_1 \rangle V_1 - \langle U_4 , V_2 \rangle V_2 - \langle U_4 , V_3 \rangle V_3 \right\|}$$

الخطوة الخامسة :-

وبنفس الطريقة نحصل على

$V_5 , V_6 , \ldots\ldots, V_n$

وبشكل عام يصبح القانون الذي يتم من خلاله ايجاد V_n

$$V_n = \frac{U_n - \langle U_n , V_1 \rangle V_1 - \ldots\ldots\ldots - \langle U_n , V_{n-1} \rangle V_{n-1}}{\left\| U_n - \langle U_n , V_1 \rangle V_1 - \ldots\ldots\ldots\ldots - \langle U_n , V_{n-1} \rangle V_{n-1} \right\|}$$

Ex:- using (**Gram – Schmidt Process**) to find **orthonormal basis** from the basis $S = \{ U_1 , U_2 , U_3 \}$ such that $U_1 = (0 , 1 , 0) , U_2 = (0 , 0 , 1) , U_3 = (1 , 1 , 0)$

Sol :-

لكي نحصل على قاعدة معيارية(orthonormal basis) يجب ان نجد المتجهات V_1 , V_2 , V_3 بحيث تكون جميعها **متعامدة** وكذلك طول كل واحد منها يساوي **واحد** وحسب خطوات **كرام – شميدت** يتم ذلك بالطريقة التالية

(1)

$$V_1 = \frac{U_1}{\| U_1 \|}$$

$$\| U_1 \| = 0^2 + 1^2 + 0^2 = 1$$

$$V_1 = \frac{(0 , 1 , 0)}{1} = (0 , 1 , 0)$$

(2)

$$V_2 = \frac{U_2 - \langle U_2 , V_1 \rangle V_1}{\| U_2 - \langle U_2 , V_1 \rangle V_1 \|}$$

Now $U_2 - \langle U_2 , V_1 \rangle V_1 = (0 , 0 , 1) – ((0 , 0 , 1) (0 , 1 , 0)) (0 , 1 , 0)$

$= (0,0,1) - (0+0+0)(0,1,0)$
$= (0,0,1) - 0 = (0,0,1)$

$|| U_2 - < U_2 , V_1 > V_1 || \quad = \quad 0^2 + 0^2 + 1^2 \quad = 1$

There fore

$V_2 = (0,0,1)$

(3) to find V_3

$$V_3 = \frac{U_3 - < U_3 , V_1 > V_1 - < U_3 \ V_2 > V_2}{|| U_3 - < U_3 , V_1 > V_1 - < U_3 , V_2 > V_2 ||}$$

Then $U_3 - < U_3 , V_1 > V_1 - < U_3 , V_2 > V_2 =$

$= (1,1,0) - ((1,1,0)(0,1,0)(0,1,0) - ((1,1,0)(0,0,1))(0,0,1)$

$= (1,1,0) - ((0+1+0))(0,1,0) - ((0+0+0)(0,0,1))$

$= (1,1,0) - (0,1,0) - 0$

$= (1,0,0)$

$|| U_3 - < U_3 , V_1 > V_1 - < U_3 , V_2 > V_2 ||$

$= \quad 1^2 + 0^2 + 0^2 \quad = 1$

Thus $V_3 = (1,0,0)$

$S^* = \{ V_1 , V_2 , V_3 \}$ is orthonormal basis

<u>Ex:-</u> using (**Gram – Schmidt Process**) to find **orthonormal basis** from the basis

(a) $S = \{ (2,0,0), (0,1,1), (0,1,-1) \}$

(b) $S = \{ (1,1,1), (-1,0,-1), (-1,2,3) \}$

(c) $S = \{ (1,1,1), (1,1,0), (1,0,0) \}$

Chapter Six

S_1-- Linear Transformations التحويلات الخطية

Def :- let V and U be vector spaces over the field K . if L : V ⟹ U is function from V to U , then we say that L is ***linear transformation*** if
(1) for all U , V ϵ V ⟹ L (U + V) = L (U) + L (V)
(2) for k ϵ K , U ϵ V ⟹ L (k U) = k L (U)

ملاحظة :-
ان عملية الجمع U + V خاصة بالفضاء V بينما عملية الجمع L (U) + L (V) هي خاصة بالفضاء U وكذلك عملية الضرب .

Ex:- show that L : R^3 ⟹ R^2 be defined by L (U_1 , U_2 , U_3) = (U_1 , U_2) is ***linear transformation***

Sol :-
(1) let U , V ϵ R^3 ⟹ U = (U_1 , U_2 , U_3) and V = (V_1 , V_2 , V_3) by def

L (U + V) = L ((U_1 , U_2 , U_3) + (V_1 , V_2 , V_3))
= L (($U_1 + V_1$, $U_2 + V_2$, $U_3 + V_3$)) (by sum of vector)
= (($U_1 + V_1$) , ($U_2 + V_2$)) (by def of L)
= (U_1 , U_2) + (V_1 , V_2)
= L (U_1 , U_2 , U_3) + L (V_1 , V_2 , V_3)
= L (U) + L (V)

(2) let kϵ R
L (k U) = L (k (U_1 , U_2 , U_3))
= L (k U_1 , k U_2 , k U_3)
= (k U_1 , k U_2)
= k (U_1 , U_2)
= k L (U_1 , U_2 , U_3)
= k L (U)

by (1) and (2)
therefore , **L** is ***linear transformation***

Ex :- (2) Let L : R^3 ⟹ R^2 be defined by

L (U_1 , U_2 , U_3) = (U_1 , $U_1 + U_2 + U_3$) . show that L is linear transformation

Sol :- (1) let U , V ϵ R^3

L (U + V) = L ((U_1 , U_2 , U_3) + (V_1 , V_2 , V_3))
= L ($U_1 + V_1$, $U_2 + V_2$, $U_3 + V_3$)
= (($U_1 + V_1$) , ($U_1 + V_1$) + ($U_2 + V_2$) + ($U_3 + V_3$))
= ($U_1 + V_1$, ($U_1 + U_2 + U_3$) + ($V_1 + V_2 + V_3$))
= (U_1 , $U_1 + U_2 + U_3$) + (V_1 , $V_1 + V_2 + V_3$)
= L (U_1 , U_2 , U_3) + L (V_1 , V_2 , V_3)
= L (U) + L (V)

(2) Let $K \in R$ and $U \in R^3$

$L(K(U)) = L(KU_1, KU_2, KU_3)$

$= KU_1, KU_1 + KU_2 + KU_3$

$= K(U_1, U_1 + U_2 + U_3)$

$= KL(U)$

by (1) and (2) there for L is linear transformation

Ex :- (3) let $T: R^2 \Longrightarrow R^2$ be defined by $T(x, y) = (x, y + 1)$. determine whether T is linear transformation

Sol :-

(1) let $V, U \in R^2 \Longrightarrow T(V) = (V_1, V_2 + 1)$

$T(U) = (U_1, U_2 + 1)$

$T(V + U) = T((V_1, V_2) + (U_1, U_2))$

$= T((V_1 + U_1, V_2 + U_2))$

$= (V_1 + U_1, V_2 + U_2 + 1)$

but $T(V) + T(U) = (V_1 + U_1, V_2 + U_2 + 2)$

$T(V + U) \neq T(V) + T(U)$

Then T is not linear transformation

S_2-- The Kernal and Rang Of Linear Transformation
نواة ومدى التحويلات الخطية

Def :- let $L: V \longrightarrow W$ be a linear transformation the ***kernel*** of L is the sub set of V consisting of all vectors V such that $L(v) = O_W$ and denoted by ker(L)

$Ker(L) = \{ v \in V : L(v) = O_W \}$

(النواة:- مجموعة كل العناصر في V بحيث صورتها تساوي المتجه الصفري في W تحت تاثير الدالة الخطية L)

Ex :- if $L: R^3 \longrightarrow R^2$ be defined by $L(U_1, U_2, U_3) = (U_1, U_3)$ find ker.(L)

Sol :-

$Ker(L) = \{ U \in R^3 : L(U) = O_{R^2} \}$

$= \{ (U_1, U_2, U_3) : T(U_1, U_2, U_3) = (0, 0) \}$

$= \{ (U_1, U_2, U_3) : U_1 = 0, U_2 = 0 \}$

$= \{ ((0, 0, U_3) \in R^3, U_3 \in R \}$

Ex :- (2) let $T: R \longrightarrow R^2$ be defined by

$T(U) = (U, 2U)$, find ker (T)

Sol :-

$Ker(T) = \{ U \in R : T(U) = O_{R^2} \}$

$= \{ U \in R : (U, 2U) = (0, 0) \}$

$= \{ U \in R : U = 0 \}$

$= \{ 0 \}$

Ex :- (3) let $T : R^3 \longrightarrow R^3$ be defined by
$T(U_1, U_2, U_3) = (U_1 + U_2, U_2, U_1 - U_3)$, find ker (T)

Sol :-

$Ker(T) = \{ U \in R^3 : T(U) = O_{R^3} \}$
$= \{ (U_1, U_2, U_3) \in R^3 : T(U_1, U_2, U_3) = (0, 0, 0) \}$
$= \{ (U_1, U_2, U_3) \in R^3 : (U_1 + U_2, U_2, U_1 - U_3) = (0, 0, 0) \}$
$= \{ (U_1, U_2, U_3) \in R^3 : U_1 + U_2 = 0, U_2 = 0, U_1 - U_3 = 0 \}$
$= \{ (U_1, U_2, U_3) \in R^3, U_1 = 0, U_2 = 0, U_3 = 0 \}$
$= \{ (0, 0, 0) \}$

<u>Ex :-</u> (4) let $T : R^4 \longrightarrow R^2$ be defined by
$T(x, y, z, w) = (x + y, z + w)$, find ker (T)

<u>Def:-</u> let $L : V \longrightarrow W$ is a linear transformation the ***Range*** of L is the set of all vectors in W that are images under L . of vectors in V .
Range (T) = $\{ w \in W : v \in V \ \text{s.t} \ T(v) = w \}$

المدى :- مجموعة المتجهات في W والتي تكون صور لمتجهات من V تحت تأثير الدالة الخطية L

<u>Theorm :-</u> if $T : V \longrightarrow W$ is linear transformation then
(1) Ker (T) is ***subspace*** of V .
(2) Range (T) is ***subspace*** of W .

<u>Proof :- (1)</u>
(a) let $U, V \in Ker(T)$
$T(U) = 0$, $T(V) = 0$
$T(U + V) = T(U) + T(V)$ (T. linear tr.)
$= 0 + 0$
$= 0$
Then , $U + V \in Ker(T)$
(2) let $K \in R$, $U \in V$
$T(U) = 0$
$T(KU) = K\,T(U)$
$= K . 0$
$= 0$
Thus , $KU \in Ker(T)$
By (1) and (2) Ker (T) is subspace .

<u>Proof :- (2)</u>
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<u>Ex :-</u> let $T : R^2 \longrightarrow R^2$ such that
$T(0, 1) = (1, 2)$, $T(1, 1) = (2, -2)$, then find $T(3, -2)$ and find $T(a, b)$?
Such that $S = ((0, 1), (1, 1))$ is spans of R^2

Sol :-

S spans R^2

Then , every vector in R^2 is linear comb . of element of S .

$K_1 (0 , 1) + K_2 (1 , 1) = (3 , -2)$

$(0 , K_1) + (K_2 , K_2) = (2 , -2)$

$(K_2 , K_1 + K_2) = (3 , -2)$

$K_2 = 3$

$K_1 + K_2 = -3 \Longrightarrow K_1 = -5$

$(3 , -2) = -5 (0 , 1) + 3 (1 , 1)$

$T (3 , -2) = T (-5 (0 , 1) + 3 (1 , 1))$

$= T (-5 (0 , 1)) + T (3 (1 , 1))$

$= -5 T (0 , 1) + 3 T (1 , 1)$

$= -5 (1 , 2) + 3 (2 , -3)$

$= (-5 , -10) + (6 , -9)$

$= (1 , -19)$

(2) to find T (a , b)

$C_1 (0 , 1) + C_2 (1 , 1) = (a , b)$

بعد حل هذه المعادلة نحصل على

$C_1 = b - a , \ C_2 = a$

Then

$(a , b) = (b - a) (0 , 1) + a (1 , 1)$

$T (a , b) = T ((b - a) (0 , 1) + a (1 , 1))$

$= T (b - a) (0 , 1) + T (a (1 , 1))$

$= (b - a) T (0 , 1) + a T (1 , 1))$

$= (b - a) (1 , 2) + a (2 , -3)$

$= (b - a , 2 (b - a) + (2 a , -3 a)$

$= (b + a , 2 b - 5 a)$

Exc:- if $T : R^2 \longrightarrow R^2$ is linear trans . and $T (1 , 0) = (2 , -2) , T (0 , 1) = (4 , 1)$ find $T (3 , 2)$ and $T (a , b)$?

S_3- Matrix Of Linear Transformation مصفوفة التحويل

ملاحظة :- لايجاد مصفوفة التحويل (***Matrix Of Linear Tran .***) ذات السعة $m * n$ بحيث $T (U) = A U$ حيث

$T : R^n \Longrightarrow R^m$

نفرض ان $e_1 , e_2 , \ldots\ldots , e_n$ قاعدة اعتيادية فان

$T (e_1) , T (e_2) , \ldots\ldots , T (e_n)$

تمثل اعمدة المصفوفة A .

Ex : - let $T = R^2 \Longrightarrow R^2$ is L . transformation such that

$$T \begin{pmatrix} x_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} x + 2 x_2 \\ x_1 - x_2 \end{pmatrix}$$

Find a matrix A such that $T (x) = A x$.

Sol :-

Let S = ((1 , 0) , (0 , 1)) be a basis of R^3 then

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$$

$$A = \begin{pmatrix}1 & 2\\1 & -1\end{pmatrix}$$

<u>Ex :</u>- find (***Matrix Of Linear Tran*** .) of $T : R^3 \Longrightarrow R^4$ defined by

$$T\begin{pmatrix}x_1\\x_2\\X_3\end{pmatrix} = \begin{pmatrix}x_1 + 2x_2\\3x_1 - x_2\\x_2 - x_3\\x_1\end{pmatrix}$$

<u>Sol :-</u> let S = { (1 , 0 , 0) , (0 , 1 , 0) , (0 , 0 , 1) } be a basis of R^3
Then

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\3\\0\\1\end{pmatrix}, \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\\1\\0\end{pmatrix}, \quad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\-1\\0\end{pmatrix}$$

اذن حسب تعريف مصفوفة التحويل الخطي تصبح مجموعة الصور لدالة T اعمده للمصفوفة A (مصفوفة التحويل الخطي T)

$$A = \begin{pmatrix}1 & 2 & 0\\3 & -1 & 0\\0 & 1 & -1\\1 & 0 & 0\end{pmatrix}_{4*3}$$

Chapter Seven

S_1--- Eigenvalues and Eigenvectors

القيم الذاتية والمتجهات الذاتية

Def :-Let **A** be a square matrix ,then the real number λ is called ***Eigen value*** if there exist a nonzero vector X in R^n such that

$A X = \lambda X$ ………………(1)
$\lambda X - A \ X = 0$
$(\lambda I_n - A) X = 0$ ……………….(1)

ملاحظة :- كل متجه غير صفري X يحقق المعادلة (1) يسمى بالمتجه الذاتي *Eigen vector* للمصفوفة A المرافق (المرتبط) بالقيمة الذاتية λ

ملاحظة :- لتكن A مصفوفة ذات سعة (n* n) يقال للمحدد ($|\lambda I_n - A|$) بمتعددة الحدود المميزة Characteristic Polynomial للمصفوفة A بينما تسمى المعادلة $|\lambda I_n - A| = 0$ المعادلة المميزة Characteristic equation للمصفوفة A

Ex :- Find the Eigen value and Eigen vector of the matrix

جد القيم الذاتية والمتجهات الذاتية للمصفوفة

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

الحل :- نطبق المعادلة المميزة ثم ناخذ لها محدد (هذه الخطوة ثابتة في كل الامثلة)

$$\lambda I_2 - A = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{pmatrix}$$

$$\left| \lambda I_2 - A \right| = 0$$

$$\begin{vmatrix} \lambda -1 & -1 \\ 2 & \lambda - 4 \end{vmatrix} = 0$$

$(\lambda - 1) (\lambda - 4) + 2 = 0$

$\lambda^2 - 5 \lambda + 4 + 2 = 0$

$\lambda^2 - 5 \lambda + 6 = 0$

$(\lambda - 3) (\lambda - 2) = 0$

either $\lambda - 3 = 0 \Longrightarrow \lambda = 3$

or $\lambda - 2 = 0 \Longrightarrow \lambda = 2$

اذن القيم الذاتية Eigen value للمصفوفة A هي $\lambda_1 = 3$, $\lambda_2 = 2$

الان لايجاد قيمة المتجه الذاتي X نستخدم قيم *h* التي حصلنا عليها ونعوضها في المعادله التالية

(1) If $\lambda_1 = 3$

$(\lambda_1 I_2 - A) X = 0$ ……………………….(1)

$(3 I_2 - A) X = 0$

$$\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}_{2*2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{2*1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$2 x_1 - x_2 = 0$

$2 x_1 - x_2 = 0$

……………………………

$2 x_1 = x_2$

let $x_2 = r$, $r \in R$

$x_1 = r / 2$

$$X_1 = \begin{pmatrix} r/2 \\ r \end{pmatrix}$$

x_1 المتجه الذاتي الاول المرتبط (المرافق) للقيمة الذاتية $\lambda_1 = 3$

(2)If $\lambda_2 = 2$

$(\lambda_2 I_2 - A) X = 0$ ……………….(1)

$(2 I_2 - A) X = 0$

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1 - x_2 = 0$

$2x_1 - 2x_2 = 0$

$x_1 = x_2$

Let $x_2 = s$ $\quad s \in R$

$s_1 = s$

$$X_2 = \begin{pmatrix} s \\ s \end{pmatrix} = S \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

المتجه الذاتي الثاني المرتبط (المرافق) للقيمة الذاتية X_2 $\lambda = 2$

Ex :- Find the Eigen value and Eigen vector of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{pmatrix}$$

الحل :-

$$\lambda I_3 - A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{pmatrix} = \begin{pmatrix} \lambda - 1 & 0 & 0 \\ 1 & \lambda - 3 & 0 \\ -3 & -2 & \lambda + 2 \end{pmatrix}$$

$$\left| \lambda I_3 - A = 0 \right|$$

$(\lambda - 1)(\lambda - 3)(\lambda + 2) = 0$

either $\lambda - 1 = 0 \Rightarrow \lambda = 1$

or $\lambda - 3 = 0 \Rightarrow \lambda = 3$

or $\lambda + 2 = 0 \Rightarrow \lambda = -2$

this is the Eigenvalue of the matrix A

هذه هي القيم الذاتية

الان نجد المتجهات الذاتية المرافقةالى λ

(1) If $\lambda = 1$

$(I_3 - A)X = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \\ -3 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2x_2$

$-3x_1 - 2x_2 + 3x_3 = 0 \Rightarrow 3x_3 = 3x_1 + 2x_2 = 6x_2 + 2x_2 = 8x_2$

$x_3 = 8/3\, x_2$

let $x_2 = r$

$$X_1 = \begin{pmatrix} 2r \\ r \\ 8/3\, r \end{pmatrix}$$

بما أن r هو عدد اختياري ينتمي الحقل الاعدادالحقيقية (ليكن r = 3)

if $r = 3$

$$X_1 = \begin{pmatrix} 6 \\ 3 \\ 8 \end{pmatrix}$$

this is the Eigenvector of the matrix A with $\lambda = 1$ المتجه الذاتي المرافق للقيمة الذاتية

(2)If $\lambda = 3$

$(3I_3 - A)X = 0$

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$2x_1 = 0$

$x_1 = 0$

$-3x_1 - 2x_2 + 5x_3 = 0$

....

$-2x_2 + 5x_3 = 0 \Rightarrow -2x_2 = -5x_3 \Rightarrow x_2 = 5/2\, x_3$

let $x_3 = r$

$$X_2 = \begin{pmatrix} 0 \\ 5/2\, r \\ r \end{pmatrix}$$

بما أن r هو عدد اختياري ينتمي الحقل الاعداد الحقيقية (ليكن r = 2)

if $r = 2$

$$X_2 = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$$

this is the Eigen vector of the matrix A with $\lambda = 3$ المتجه الذاتي المرافق للقيمة الذاتية

(3) If $\lambda = -2$

$(-2 I_3 - A) X = 0$

$$\begin{pmatrix} -3 & 0 & 0 \\ 1 & -5 & 0 \\ -3 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-3 x_1 = 0$

$x_1 - 5 x_2 = 0$

$-3 x_1 - 2 x_2 = 0$

…………………………………

$x_1 = x_2 = 0$

$$X_3 = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$

بما أن r هو عدد اختياري ينتمي الحقل الاعدادالحقيقية (ليكن r = 1)

if r = 1

$$X_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

this is the Eigenvector of the matrix A with $\lambda = -2$ المتجه الذاتي المرافق للقيمة الذاتية

<u>**Exc :-**</u> Find the Eigenvalues and Eigenvector of the matrix

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

S_{2}- Cayley- Hamilton Theorem (كايلي - هاميلتون)

نظرية:- كل مصفوفة مربعة A تحقق معادلتها المميزة .

مثال :- هل ان المصفوفة A تحقق معادلتها المميزة

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

الحل:-

$$\left| h \; I_2 - A \right| = 0$$

$$h \; I_2 - A = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} - \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda - 5 & -2 \\ -2 & \lambda - 1 \end{pmatrix}$$

$$\left| \lambda I_2 - A \right| = 0$$

$$(\lambda - 5)(\lambda - 1) - 4 = 0$$

$$\lambda^2 - 6\lambda + 5 - 4 = 0$$

$$\lambda^2 - 6\lambda + 1 = 0$$

نعوض بدل كل λ بالمصفوفه A ونضرب الحد الخالي من λبمصفوفة الوحده

$$A^2 - 6A + I = 0$$

الان نجد A^2

$$A^2 = A \cdot A$$

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 29 & 12 \\ 12 & 5 \end{pmatrix}$$

$$A^2 - 6A + I_2 = 0$$

$$\begin{pmatrix} 29 & 12 \\ 12 & 5 \end{pmatrix} - \begin{pmatrix} 30 & 12 \\ 12 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

اذن تحقق المعادلة المميزة

Ex:- if A be square matrix find A^{-1} ,by using **Cayley- Hamilton Theorem** ,where

$$A = \begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix}$$

Sol:-

$$\left| \lambda I_2 - A \right| = 0$$

$$\left| \lambda I_2 - A \right| = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda -3 & 2 \\ -1 & \lambda -2 \end{pmatrix}$$

$$\left| \lambda I_2 - A \right| = 0$$

$(\lambda-3)(\lambda - 2) + 2 = 0$

$\lambda^2 - 5\lambda + 6 + 2 = 0$

$\lambda^2 - 5\lambda + 8 = 0$

نعوض بدل كل λ بالمصفوفه A ونضرب الحد الخالي من λ بمصفوفةالوحده

$A^2 - 5A + 8I_2 = 0$

$A^2 - 5A = -8I_2$

$(A - 5I_2)A = -8I_2$

$-1/8\,(A - 5I_2)A = I_2$

الان نضرب الطرفين ب A^{-1}

$A^{-1} = -1/8\,(A - 5I_2)$

$$A - 5I_2 = \begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 1 & -3 \end{pmatrix}$$

$A^{-1} = -1/8\,(A - 5I_2)$

$$= -1/8 \begin{pmatrix} -2 & -2 \\ 1 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1/4 & 1/4 \\ -1/8 & 3/8 \end{pmatrix}$$

Ex:- if A be square matrix find A^{-1} ,by using **Cayley- Hamilton Theorem** ,where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -3 \\ 2 & 2 & 4 \end{pmatrix}$$

H. W

S_3-- Similar Of Matrices تشابه المصفوفات

Def :- the matrix B is called ***Similar*** to the matrix A if there exist a matrix P has inverse such that $B = P^{-1} A P$

Ex:- Let

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Find B such that ***similar*** to A

Sol:-

$B = P^{-1} A P$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

تمثل المصفوفة B