S2.1 : Functions and Their Graphs

Definition: A function \( f \) (or a mapping \( f \)) from a set \( A \) to a set \( B \) is a rule that assigns to each element \( a \) of \( A \) exactly one element \( b \) of \( B \). The set \( A \) is called the domain of \( f \) and the set \( B \) is called the codomain of \( f \). If \( f \) assigns \( b \) to \( a \), then \( b \) is called the image of \( a \) under \( f \). The subset of \( B \) comprised of all the images of elements of \( A \) under \( f \) (which is denoted by \( f(A) \)) is called the image of \( A \) under \( f \) (or the range of \( f \)).

We use \( f : A \to B \) to mean that \( f \) is a function from \( A \) to \( B \). We will write \( f(a) = b \) to indicate that \( b \) is the image of \( a \) under \( f \).

Example 2.1.1:
Let \( A = \{2, 4, 5\} \), \( B = \{1, 2, 3, 6\} \), and \( f : A \to B \) be the function defined by \( f(2) = 1, f(4) = 3, f(5) = 6 \). Then the domain of \( f \) is \( A = \{2, 4, 5\} \), the codomain of \( f \) is \( B = \{1, 2, 3, 6\} \), and the range of \( f \) is \( \{1, 3, 6\} \).

Counter example:
Let \( C = \{1, 2, 3, 4\} \) and \( D = \{2, 3, 4, 5\} \), and let \( h \) be the rule defined by \( h(1) = 2, h(1) = 4, h(2) = 3, h(3) = 5, h(4) = 4 \), then \( h \) is not a function from \( C \) to \( D \) since there are two different elements (2 and 4) belong to \( D \) are assigned to the same element 1 of \( C \).

Example 2.1.2: Find the domain and the range of the function \( f \) defined by \( f(x) = \sqrt{x+10} \).

Solution: For \( y = f(x) = \sqrt{x+10} \) to be real, \( x+10 \) must be greater than or equal to 0. That is, \( x+10 \geq 0 \) which means that \( x \geq -10 \). Thus the domain is \( \{x : x \geq -10\} \) and the range is \( \{y : y \geq 0\} \).

Exercises:

1) Let \( A = \{2, 4, 5, 7\} \), \( B = \{1, 2, 3, 6, 9\} \), and \( f : A \to B \) be the function defined by \( f(2) = 9, f(4) = 3, f(5) = 6, f(7) = 2 \). Find the domain of \( f \), the codomain of \( f \), and the range of \( f \).
2) Let \( f \) be a function defined by \( f(x) = \frac{1}{x+2} \). Find the domain and the range of the function \( f \).

3) Find the domain and the range of the function \( f \) defined by \( f(x) = \sqrt{2x-9} \).

**Definition:** The graph of a function \( f \) is the line passing through all the points \((x, f(x))\) on the \( xy \)-plane.

**Definition:** The \( y \)-coordinate of the point where a graph of a function intersect the \( y \)-axis is called the \( y \)-intercept of the function.

**Definition:** The \( x \)-coordinate of a point where a graph of a function intersects the \( x \)-axis is called an \( x \)-intercept of the function.

**Remarks:**

1) The graph of any function \( f \) has at most one \( y \)-intercept. The graph of the function \( f \) has exactly one \( y \)-intercept if 0 is in the domain of the function \( f \) and the \( y \)-intercept is \( f(0) \).

2) The graph of any function \( f \) has no \( x \)-intercept if there is no \( x \) in the domain of the function \( f \) such that \( f(x) = 0 \).

The graph of a function \( f \) has one or more than one \( x \)-intercepts if \( f(x) = 0 \) for some \( x \) in the domain of \( f \), and the number of \( x \)-intercepts is the number of the distinct solutions of the equation \( f(x) = 0 \).

**Properties of Functions:**

1) A function \( y = f(x) \) is called an even function of \( x \) if \( f(-x) = f(x) \), \( \forall x \).

2) A function \( y = f(x) \) is called an odd function of \( x \) if \( f(-x) = -f(x) \), \( \forall x \).

**S2.2 : Linear Functions and their Graphs**

**Definition:** A function \( f : R \rightarrow R \) is called a linear function if \( f \) is defined by \( f(x) = ax + b \), \( a \neq 0 \)

where \( a \) and \( b \) are real numbers.
Example 2.2.1: The function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 3x + 12$ is a linear function.

Example 2.2.2: The function $g: \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = x - 0.2$ is a linear function.

Example 2.2.3: The function $h: \mathbb{R} \rightarrow \mathbb{R}$ defined by $h(x) = -\frac{3}{2}x + 1$ is a linear function.

Example 2.2.4: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the linear function defined by $f(x) = 4x + 10$. Find the $x$-intercept and the $y$-intercept of $f$.

Solution: $f(x) = 0 \Rightarrow 4x + 10 = 0$  
$\quad \quad \quad \quad \Rightarrow 4x = -10$  
$\quad \quad \quad \quad \Rightarrow x = -\frac{10}{4} = -2.5$

Therefore the $x$-intercept is $-2.5$

$f(0) = 10 \Rightarrow$ the $y$-intercept is $10$.

Example 2.2.5: Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be the linear function defined by $g(x) = \frac{1}{5}x - 6$. Find the $x$-intercept and the $y$-intercept of $g$.

Solution: $g(x) = 0 \Rightarrow \frac{1}{5}x - 6 = 0$  
$\quad \quad \quad \quad \Rightarrow \frac{1}{5}x = 6 \Rightarrow x = 30$

Therefore the $x$-intercept is $30$

$g(0) = -6 \Rightarrow$ the $y$-intercept is $-6$.

**Graph of a linear function:**

The graph of a linear function $f$ is the straight line passing through the two points $(a, 0)$ and $(0, b)$ where $a$ is the $x$-intercept of the function $f$ and $b$ is the $y$-intercept of the function $f$.

**Remark:** The graph of any linear function $f$ has exactly one $x$-intercept and has exactly one $y$-intercept.
Example 2.2.6: Let $f: \mathbb{R} \to \mathbb{R}$ be the linear function defined by $f(x) = -2x + 7$. Find the $x$-intercept and the $y$-intercept of $f$, then graph the function $f$.

Solution: 

$$f(x) = 0 \Rightarrow -2x + 7 = 0$$

$$\Rightarrow -2x = -7$$

$$\Rightarrow x = \frac{-7}{-2} = 3.5$$

Therefore the $x$-intercept is $3.5$.

$$f(0) = 7 \Rightarrow \text{the } y\text{-intercept is } 7.$$  

Thus the graph of the function $f$ is the straight line passing through the two points $(3.5, 0)$ and $(0, 7)$.

Thus the graph of the function $f$ is the following graph

![Graph of f](image)

Example 2.2.7: Let $g: \mathbb{R} \to \mathbb{R}$ be the linear function defined by $g(x) = 4x + 12$. Find the $x$-intercept and the $y$-intercept of $g$, then graph the function $g$.

Solution: 

$$g(x) = 0 \Rightarrow 4x + 12 = 0$$

$$\Rightarrow 4x = -12$$

$$\Rightarrow x = \frac{-12}{4} = -3$$

Therefore the $x$-intercept is $-3$.

$$g(0) = 12 \Rightarrow \text{the } y\text{-intercept is } 12.$$
Thus the graph of the function $g$ is the straight line passing through the two points $(-3, 0)$ and $(0, 12)$.

Thus the graph of the function $g$ is the following graph

![Graph of the function $g$](image)

**Exercises:**

1) Let $f : \mathbb{R} \to \mathbb{R}$ be the linear function defined by $f(x) = 3x - 10$. Find the $x$-intercept and the $y$-intercept of $f$.

2) Let $g : \mathbb{R} \to \mathbb{R}$ be the linear function defined by $g(x) = 0.3x + 0.7$. Find the $x$-intercept and the $y$-intercept of $g$.

3) Let $f : \mathbb{R} \to \mathbb{R}$ be the linear function defined by $f(x) = -4x + 8$. Find the $x$-intercept and the $y$-intercept of $f$, then graph the function $f$.

4) Let $g : \mathbb{R} \to \mathbb{R}$ be the linear function defined by $g(x) = 5x + 15$. Find the $x$-intercept and the $y$-intercept of $g$, then graph the function $g$.

**S2.3: Some well-known Functions and their Graphs**

1) A function $f(x) = c$ where $c$ is a fixed number is called a constant function.
Example 2.3.1: The function \( y = f(x) = 1 \) is a constant function and its graph is

2) The absolute value function \( y = f(x) = |x| \) is defined by the formula

\[
y = f(x) = |x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0
\end{cases}
\]

and its graph is

Remember that \( |x| = \sqrt{x^2} \).

3) A function \( y = f(x) = x^r \) where \( r \) is a real number is called a power function.

Example 2.3.2:

The function \( y = f(x) = x^2 \) is a power function (which is also a quadratic function) and its graph is
Example 2.3.3: The function $y = f(x) = x^3$ is a power function and its graph is

Example 2.3.4: The function $y = f(x) = \sqrt{x}$ is a power function and its graph is
Example 2.3.5: The function \( y = f(x) = \frac{1}{x} \) is a power function and its graph is

![Graph of \( \frac{1}{x} \)](image)

4) Let \( a \) be a positive real number other than 1. The function \( y = f(x) = a^x \) is called the exponential function with base \( a \).

Example 2.3.6: Graph the exponential function \( y = 2^x \)

Answer: To draw the graph of \( y = 2^x \), we can make use of a table giving values for \( x \) and find the corresponding values for \( y \)

\[
\begin{align*}
x = 0 & \quad \text{gives} \quad y = 2^0 = 1, \\
x = 1 & \quad \text{gives} \quad y = 2^1 = 2, \\
x = -1 & \quad \text{gives} \quad y = 2^{-1} = \frac{1}{2}.
\end{align*}
\]

Following the process we make the table

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2^x )</td>
<td>0.0625</td>
<td>0.125</td>
<td>0.25</td>
<td>0.5</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
</tr>
</tbody>
</table>
Example 2.3.7: The function $y = 5^x$ is an exponential function and its graph is

Answer:

$x = 0$ gives $y = 5^0 = 1$,

$x = 1$ gives $y = 5^1 = 5$,

$x = -1$ gives $y = 5^{-1} = 0.2$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5^x$</td>
<td>0.04</td>
<td>0.2</td>
<td>1</td>
<td>5</td>
<td>25</td>
</tr>
</tbody>
</table>

Exercise 2.3.8: Graph the exponential function $y = 10^x$.

The properties of exponential function and their graph

- The domain is $\mathbb{R}$ (set of real numbers).
- The range is $\mathbb{R}^+$ (set of positive real numbers).
- The graph is always continuous (no break in the graph).
Rules of Exponents: If \( a > 0 \) and \( b > 0 \), the following rules of exponent should be hold for all real numbers \( x \) and \( y \):

1. \( a^x \times a^y = a^{x+y} \)
2. \( \frac{a^x}{a^y} = a^{x-y} \)
3. \( a^0 = 1 \)
4. \( \frac{1}{a^x} = a^{-x} \)
5. \( (a^x)^y = (a^y)^x = a^{xy} \)
6. \( (ab)^x = a^x b^x \)
7. \( \left( \frac{a}{b} \right)^x = \frac{a^x}{b^x} \)

5) The function \( y = e^x \) is called the natural exponential function whose base is \( e \approx 2.718281828 \), and its graph is

<table>
<thead>
<tr>
<th>( x )</th>
<th>( e^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0.1353</td>
</tr>
<tr>
<td>-1</td>
<td>0.3679</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2.718</td>
</tr>
<tr>
<td>2</td>
<td>7.389</td>
</tr>
</tbody>
</table>

Remark: Graph of \( e^x \) and \( e^{-x} \) are reflections of each other.

6) The function \( y = \log_b x \) is called the logarithm function with base \( b \) where \( b \) is a positive number \( \neq 1 \); and \( x > 0 \), and the graph of \( y = \log_b x \) where \( b \) is greater than 1 is the following graph.
Remark: $y = \log_b x$ means that $x = b^y$.

Example 2.3.9: The function $y = \log_2 x$ is a logarithm function with base 2 and its graph is

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.25</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \log_2 x$</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

![Graph of $y = \log_2 x$](image)

Example 2.3.10: Draw the graph of $\log_{10} x$.

Answer:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.5</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \log_{10} x$</td>
<td>-0.301</td>
<td>0</td>
<td>0.699</td>
<td>1</td>
<td>1.176</td>
<td>1.301</td>
<td>1.699</td>
<td>2</td>
</tr>
</tbody>
</table>

![Graph of $y = \log_{10} x$](image)

Rules of logarithm: For $x > 0$ and $y > 0$, and $b$ is a positive number $\neq 1$ we have the following rules:

1. $\log_b x y = \log_b x + \log_b y$
2. $\log_b \frac{x}{y} = \log_b x - \log_b y$
3. $\log_b x^y = y \cdot \log_b x$
4. $\log_b a = \frac{\log_c a}{\log_c b}$, where $c$ can be any base.
Remarks:

- The logarithm of any number to the base of the same number will be 1 (\(\log_b b = 1\), \(\log_5 5 = 1\) etc ...).
- Logarithm of 1 to any base is 0 (\(\log_b 1 = 0\), \(\log_5 1 = 0\) etc ...).
- The logarithm function is defined only for positive numbers.
- The domain of the logarithm function is \(\mathbb{R}^+\).
- The range of the logarithm function is \(\mathbb{R}\).

7) The logarithm function with base \(e\) is called the natural logarithm function and will be denoted by \(y = \ln x\) (i.e. \(y = \log_e x = \ln x\)) and its graph is

![Graph of \(y = \ln x\)]

Remarks:
- \(\ln e = 1\) (since \(\ln e = \log_e e\))
- \(\ln 1 = 0\)

Exercise 2.3.12: Draw the graph for the following logarithmic functions:
1. \(\log_5 x\)
2. \(\log_8 x\)
3. \(\log_3 x\)

8) A polynomial function is defined as
\[y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0\] where
\(a_0, a_1, \ldots, a_{n-1}, a_n\) are constants.
Example 2.3.13: The function $y = x^2 - 5x + 6$ is a polynomial function.

Algebra of Functions

Definition: The sum, difference, product, and quotient of the functions $f$ and $g$ are the functions defined by

$$(f + g)(x) = f(x) + g(x)$$

sum function

$$(f - g)(x) = f(x) - g(x)$$

difference function

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

product function

$$
\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad g(x) \neq 0
$$

quotient function

The domain of each function is the intersection of the domains of $f$ and $g$, with the exception that the values of $x$ where $g(x) = 0$ must be excluded from the domain of the quotient function.

Definition: Let $f$ and $g$ be functions, then $f \circ g$ is called the composite of $g$ and $f$ and is defined by the equation

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set

$$D = \{ x \in \text{domain } g : g(x) \in \text{domain } f \}.$$

Example 2.3.14: Let $f$ and $g$ be the functions defined by

$f(x) = x - 7$ and $g(x) = x^2 + 5$. Find the functions $f + g$, $f - g$, $f \cdot g$, $\frac{g}{f}$, $f \circ g$, $g \circ f$ and find their domains.
Solution:

\[(f + g)(x) = f(x) + g(x) = x - 7 + x^2 + 5 = x^2 + x - 2\]
\[(f - g)(x) = f(x) - g(x) = x - 7 - x^2 - 5 = -x^2 + x - 12\]
\[(f \cdot g)(x) = f(x) \cdot g(x) = (x - 7) \cdot (x^2 + 5) = x^3 - 7x^2 + 5x - 35\]
\[\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{x^2 + 5}{x - 7}\]
\[(f \circ g)(x) = f(g(x)) = f(x^2 + 5) = x^2 + 5 - 7 = x^2 - 2\]
\[(g \circ f)(x) = g(f(x)) = g(x - 7) = (x - 7)^2 + 5\]
\[= x^2 - 14x + 49 + 5 = x^2 - 14x + 54\]

The domain of \(f\) = \(\mathbb{R}\)
The domain of \(g\) = \(\mathbb{R}\)
The intersection of the domains of \(f\) and \(g\) is \(\mathbb{R}\)
Thus the domain of each of the functions \(f + g\), \(f - g\), \(f \cdot g\), \(f \circ g\), and \(g \circ f\) is \(\mathbb{R}\).
The domain of \(\frac{g}{f}\) = \(\mathbb{R} - \{7\}\).

Remark: The domain of any polynomial function is \(\mathbb{R}\).

Example 2.3.15: Let \(f\) and \(g\) be the functions defined by \(f(x) = x + 5\) and \(g(x) = x^2 - 3\). Find \(f \circ g(x)\), \(g \circ f(x)\), \(f \circ g(3)\) and \(g \circ f(3)\).

Solution:
\[f \circ g(x) = f(g(x)) = f(x^2 - 3) = x^2 - 3 + 5 = x^2 + 2\]
\[g \circ f(x) = g(f(x)) = g(x + 5) = (x + 5)^2 - 3 = x^2 + 10x + 25 - 3 = x^2 + 10x + 22\]
\[ f \circ g(3) = (3)^2 + 2 = 9 + 2 = 11 \]
\[ g \circ f(3) = (3)^2 + 10(3) + 22 = 90 + 30 + 22 = 61 \]

Exercise 2.3.16: Let \( f \) and \( g \) be the functions defined by 
\[ f(x) = x - 4 \quad \text{and} \quad g(x) = \sqrt{x} \] . Find the functions \( f + g \), \( f - g \), \( f \cdot g \), \( \frac{f}{g} \) and find their domains.

S 2.4: Unit Circle and Basic Trigonometric Functions

Definition 1: Let \( x \) be any real number and let \( U \) be the unit circle with equation \( a^2 + b^2 = 1 \) (the centre of the circle \( U \) is the point \( O(0,0) \), and the radius of the circle \( U \) equals 1). Start from the point \( A(1,0) \) on \( U \) and proceed counterclockwise if \( x \) is positive and clockwise if \( x \) is negative around the unit circle \( U \) until an arc length of \( |x| \) has been covered. Let \( P(a,b) \) be the point at the terminal end of the arc. The measurement of the angle \( AOP \) is \( x \) radians.

If \( x \) radians = \( t^\circ \) (degrees), then the following six trigonometric functions of \( x \) are defined in terms of the coordinates of the circular point \( P(a,b) \):

1) \( y = \sin x = b = \sin (x \ \text{radians}) = \sin (t \ \text{degrees}) = \sin t^\circ \)
2) \( y = \cos x = a = \cos (x \ \text{radians}) = \cos (t \ \text{degrees}) = \cos t^\circ \)
3) \( y = \tan x = \frac{b}{a} \quad (a \neq 0) \)
\[ = \tan (x \ \text{radians}) = \tan (t \ \text{degrees}) = \tan t^\circ \]
4) \( y = \cot x = \frac{a}{b} \quad (b \neq 0) \)
\[ = \cot (x \ \text{radians}) = \cot (t \ \text{degrees}) = \cot t^\circ \]
5) \( y = \sec x = \frac{1}{a} \quad (a \neq 0) \)
   \[ = \sec (\pi \text{ radians}) = \sec (t \text{ degrees}) = \sec \theta \]

6) \( y = \csc x = \frac{1}{b} \quad (b \neq 0) \)
   \[ = \csc (\pi \text{ radians}) = \csc (t \text{ degrees}) = \csc \theta \]

**Remark 1:** Definition 1 uses the standard function notation, \( y = f(x) \), with \( f \) replaced by the name of a particular trigonometric function. For example, \( y = \cos x \) actually means \( y = \cos (x) \) and \( \cos \theta \) actually means \( \cos (\theta) \).

**Remark 2:** Remember that \( \theta = t \times \frac{\pi}{180} \text{ radians} \) and \( x \text{ radians} = (x \times \frac{180}{\pi})^\circ \).

**Theorem 1:**

For any real number \( x \) we have the following trigonometric identities:

1) \( \csc x = \frac{1}{\sin x} \).

2) \( \sec x = \frac{1}{\cos x} \).

3) \( \cot x = \frac{1}{\tan x} \).

4) \( \tan x = \frac{\sin x}{\cos x} \).

5) \( \cot x = \frac{\cos x}{\sin x} \).

6) \( \sin (-x) = -\sin (x) \).

7) \( \cos (-x) = \cos (x) \).

8) \( \tan (-x) = -\tan (x) \).

9) \( \cot (-x) = -\cot (x) \).

10) \( \sin^2 x + \cos^2 x = 1 \).

11) \( \sec^2 x = \tan^2 x + 1 \).

12) \( \csc^2 x = \cot^2 x + 1 \).
S 2.5: Graphs of Sine and Cosine Functions

2.5.1: Table for values of \( \sin x \), \( \cos x \), and \( \tan x \) for selected values of \( x \)

<table>
<thead>
<tr>
<th>Values of ( x )</th>
<th>Degrees</th>
<th>0</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radians</td>
<td></td>
<td>0</td>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{\pi}{3} )</td>
<td>( \frac{\pi}{2} )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td></td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( \cos x )</td>
<td></td>
<td>1</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>( \tan x )</td>
<td></td>
<td>0</td>
<td>( \frac{1}{\sqrt{3}} )</td>
<td>1</td>
<td>( \sqrt{3} )</td>
<td>Undefined</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Values of ( x )</th>
<th>Degrees</th>
<th>120</th>
<th>135</th>
<th>150</th>
<th>180</th>
<th>270</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radians</td>
<td></td>
<td>( \frac{2\pi}{3} )</td>
<td>( \frac{3\pi}{4} )</td>
<td>( \frac{5\pi}{6} )</td>
<td>( \pi )</td>
<td>( \frac{3\pi}{2} )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td></td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( \cos x )</td>
<td></td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( -\frac{\sqrt{3}}{2} )</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( \tan x )</td>
<td></td>
<td>-( \sqrt{3} )</td>
<td>-1</td>
<td>-( \frac{1}{\sqrt{3}} )</td>
<td>0</td>
<td>Undefined</td>
</tr>
</tbody>
</table>

**Definition:** A function \( f \) is periodic if there exists a positive real number \( p \) such that \( f(x) = f(x + p) \) for all \( x \) in the domain of \( f \). The smallest such positive number \( p \) is the period of \( f \).

**Remarks:**

1) The functions \( \sin x \), \( \cos x \), \( \sec x \), and \( \csc x \) are periodic functions with period \( 2\pi \).

2) The functions \( \tan x \) and \( \cot x \) are periodic functions with period \( \pi \).
2.5.2: The Graph of $\sin x$

The graph of the function $y = \sin x$ is the line passing through all the points $(x, \sin x)$ on the $x,y$-plane.

The graph of the function $y = \sin x$ for the interval $[0, 2\pi]$ is the line passing through the points $(0,0), \left(\frac{\pi}{6}, \frac{1}{2}\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{6}, \frac{1}{2}\right), (\pi, 0), \left(\frac{7\pi}{6}, -\frac{1}{2}\right), \left(\frac{3\pi}{2}, -1\right), \left(\frac{11\pi}{6}, -\frac{1}{2}\right)$, and $(2\pi, 0)$ which is shown in the following figure.

![Graph of sin x](image)

The graph of the function $y = \sin x$ is shown in the following figure.

![Graph of sin x](image)

The period of the function $y = \sin x$ is $2\pi$. The domain of the function $y = \sin x$ is the set of all real numbers $\mathbb{R}$.

The range of the function $y = \sin x$ is the interval $[-1, 1]$.
2.5.3: The Graph of \( \cos x \)

The graph of the function \( y = \cos x \) is the line passing through all the points \((x, \cos x)\) on the \(x\)-\(y\)-plane.

The graph of the function \( y = \cos x \) for the interval \([0, 2\pi]\) is the line passing through the points \((0, 1), (\frac{\pi}{3}, \frac{1}{2}), (\frac{\pi}{2}, 0), (\frac{2\pi}{3}, -\frac{1}{2}), (\pi, -1), (\frac{4\pi}{3}, -\frac{1}{2}), (\frac{3\pi}{2}, 0), (\frac{5\pi}{3}, \frac{1}{2})\), and \((2\pi, 1)\) which is shown in the following figure.

![Graph of \( \cos x \)](image)

The graph of the function \( y = \cos x \) is shown in the following figure.

![Graph of \( \cos x \)](image)

The period of the function \( y = \cos x \) is \(2\pi\).

The domain of the function \( y = \cos x \) is the set of all real numbers \(\mathbb{R}\).

The range of the function \( y = \cos x \) is the interval \([-1, 1]\).
2.5.4: The Graphs of $\tan x$ and $\sec x$

The graph of the function $y = \tan (x)$ is the line passing through all the points $(x, \tan x)$ on the $xy$-plane.

The graph of $y = \tan (x)$ is shown in the following figure.

![Graph of tan(x)](image)

The graph of $y = \sec (x)$ is shown in the following figure.

![Graph of sec(x)](image)
Exercise: Draw the graph of the following trigonometric functions:

1) \( y = \csc (x) \)
2) \( y = \cot (x) \)