CH2: Functions

S2.1 : Functions and Their Graphs

Definition : A function f (or a mapping f) from a set A to a set B is a rule that assigns to each element a of A exactly one element b of B . The set A is called the domain of f and the set B is called the codomain of f . If f assigns b to a , then b is called the image of a under f. The subset of B comprised of all the images of elements of A under f (which is denoted by $f(A)$) is called the image of A under f (or the range of f) .

We use $f: A \rightarrow B$ to mean that f is a function from A to B . We will write $f(\mathrm{a})=\mathrm{b}$ to indicate that b is the image of a under f .

Example 2.1.1:

Let $A=\{2,4,5\}$, $B=\{1,2,3,6\}$, and $f: A \rightarrow B$ be the function defined by $f(2)=1$, $f(4)=3$, $f(5)=6$. Then the domain of f is $A=\{2,4,5\}$, the codomain of f is $B=\{1,2,3,6\}$, and the range of $f=\{1,3,6\}$.

Counter example :

Let $C=\{1,2,3,4\}$ and $D=\{2,3,4,5\}$, and let h be the rule defined by $h(1)=2$, $h(1)=4$, $h(2)=3$, $h(3)=5$, $h(4)=4$, then h is not a function from C to D since there are two different elements (2 and 4) belong to D are assigned to the same element 1 of C.

Example 2.1.2: Find the domain and the range of the function f defined by $f(x)=\sqrt{x+10}$.

Solution : For $y=f(x)=\sqrt{x+10}$ to be real , $x+10$ must be greater than or equal to 0 . That is , $x+10 \geq 0$ which means that $x \geq -10$. Thus the domain is $\{x: x \geq -10\}$ and the range is $\{y: y \geq 0\}$.

Exercises:

1) Let $A=\{2,4,5,7\}$, $B=\{1,2,3,6,9\}$, and $f: A \rightarrow B$ be the function defined by $f(2)=9$, $f(4)=3$, $f(5)=6$, $f(7)=2$. Find the domain of f , the codomain of f , and the range of f .

2) Let f be a function defined by $f(x) = \frac{1}{x+2}$. Find the domain and the range of the function f .

3) Find the domain and the range of the function f defined by $f(x) = \sqrt{2x-9}$.

Definition: The graph of a function f is the line passing through all the points $(x, f(x))$ on the xy-plane .

Definition: The y-coordinate of the point where a graph of a function intersect the y-axis is called the y-intercept of the function .

Definition: The x-coordinate of a point where a graph of a function intersects the x-axis is called an x-intercept of the function .

Remarks :

1) The graph of any function f has at most one y-intercept . The graph of the function f has exactly one y-intercept if 0 is in the domain of the function f and the y-intercept is $f(0)$.

2) The graph of any function f has no x-intercept if there is no x in the domain of the function f such that $f(x)=0$.
The graph of a function f has one or more than one x-intercepts if $f(x)=0$ for some x in the domain of f , and the number of x-intercepts is the number of the distinct solutions of the equation $f(x)=0$.

Properties of Functions :

1) A function $y=f(x)$ is called an even function of x if $f(-x)=f(x)$, $\forall$ x .

2) A function $y=f(x)$ is called an odd function of x if $f(-x)=-f(x)$, $\forall$ x .

S2.2 : Linear Functions and their Graphs

Definition: A function $f: R \to R$ is called a linear function if f is defined by $f(x)=ax+b$, $a \neq 0$
where a and b are real numbers .

Example 2.2.1: The function $f: R \to R$ defined by $f(x) = 3x + 12$ is a linear function .

Example 2.2.2: The function $g: R \to R$ defined by $g(x) = x - 0.2$ is a linear function .

Example 2.2.3: The function $h: R \to R$ defined by $h(x) = -\frac{3}{2}x + 1$ is a linear function .

Example 2.2.4: Let $f: R \to R$ be the linear function defined by $f(x) = 4x + 10$. Find the x - intercept and the y - intercept of f .

Solution: $f(x) = 0 \Rightarrow 4x + 10 = 0$

$$\Rightarrow 4x = -10$$

$$\Rightarrow x = -\frac{10}{4} = -2.5$$

Therefore the x - intercept is -2.5

$f(0) = 10 \Rightarrow$ the y - intercept is 10 .

Example 2.2.5: Let $g: R \to R$ be the linear function defined by $g(x) = \frac{1}{5}x - 6$. Find the x - intercept and the y - intercept of g .

Solution: $g(x) = 0 \Rightarrow \frac{1}{5}x - 6 = 0$

$$\Rightarrow \frac{1}{5}x = 6 \Rightarrow x = 30$$

Therefore the x - intercept is 30

$g(0) = -6 \Rightarrow$ the y - intercept is -6 .

Graph of a linear function :

The graph of a linear function f is the straight line passing through the two points $(a, 0)$ and $(0, b)$ where a is the x - intercept of the function f and b is the y - intercept of the function f .

Remark : The graph of any linear function f has exactly one x - intercept and has exactly one y - intercept .

Example 2.2.6: Let $f : R \to R$ be the linear function defined by $f(x) = -2x + 7$. Find the x - intercept and the y - intercept of f , then graph the function f .

Solution: $f(x) = 0 \Rightarrow -2x + 7 = 0$

$$\Rightarrow -2x = -7$$

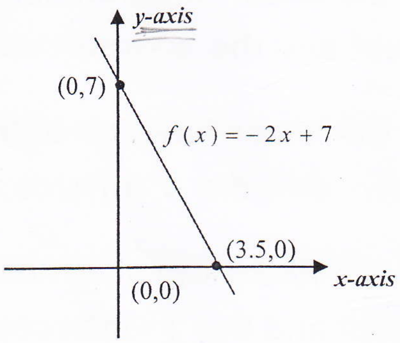
$$\Rightarrow x = \frac{-7}{-2} = 3.5$$

Therefore the x - intercept is 3.5 .

$f(0) = 7 \Rightarrow$ the y - intercept is 7 .

Thus the graph of the function f is the straight line passing through the two points $(3.5, 0)$ and $(0, 7)$.

Thus the graph of the function f is the following graph



Example 2.2.7: Let $g : R \to R$ be the linear function defined by $g(x) = 4x + 12$. Find the x - intercept and the y - intercept of g , then graph the function g .

Solution: $g(x) = 0 \Rightarrow 4x + 12 = 0$

$$\Rightarrow 4x = -12$$

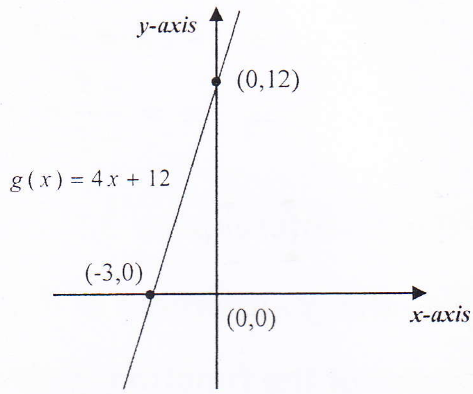
$$\Rightarrow x = \frac{-12}{4} = -3$$

Therefore the x - intercept is -3

$g(0) = 12 \Rightarrow$ the y - intercept is 12 .

Thus the graph of the function g is the straight line passing through the two points $(-3, 0)$ and $(0, 12)$.

Thus the graph of the function g is the following graph



Exercises:

1) Let $f: R \to R$ be the linear function defined by $f(x) = 3x - 10$.
   Find the x - intercept and the y - intercept of f .

2) Let $g: R \to R$ be the linear function defined by $g(x) = 0.3x + 0.7$.
   Find the x - intercept and the y - intercept of g .

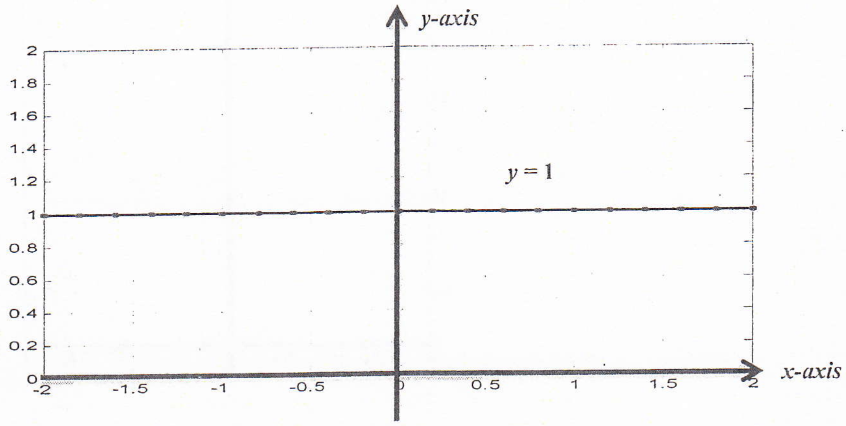
3) Let $f: R \to R$ be the linear function defined by $f(x) = -4x + 8$.
   Find the x - intercept and the y - intercept of f , then graph the function f .

4) Let $g: R \to R$ be the linear function defined by $g(x) = 5x + 15$.
   Find the x - intercept and the y - intercept of g , then graph the function g .

S2.3 : Some well-known Functions and their Graphs

1) A function $f(x) = c$ where c is a fixed number is called a constant function .

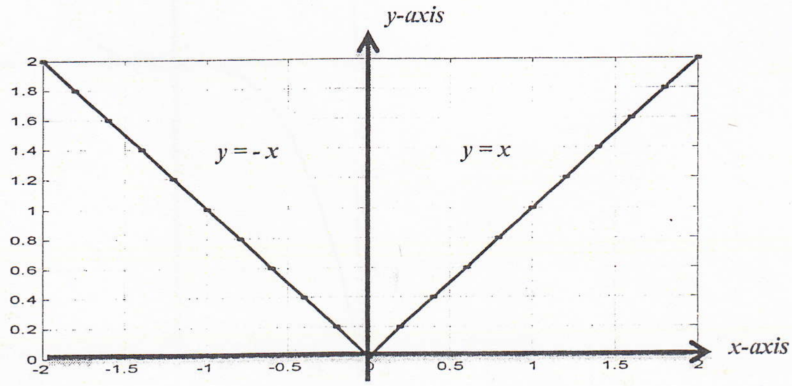
Example 2.3.1 : The function $y = f(x) = 1$ is a constant function and its graph is



2) The absolute value function $y = f(x) = |x|$ is defined by the formula

$$y = f(x) = |x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

and its graph is

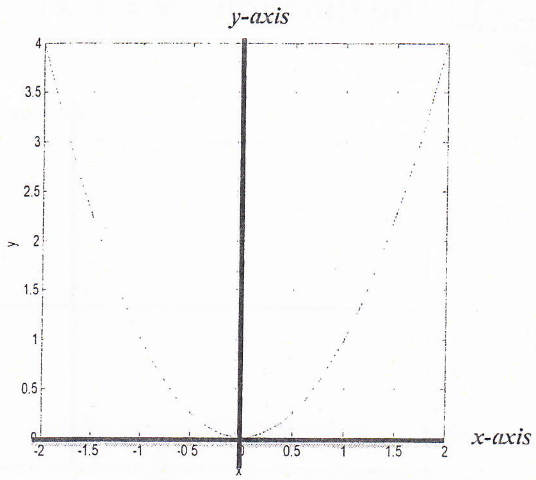


Remember that $|x| = \sqrt{x^2}$.

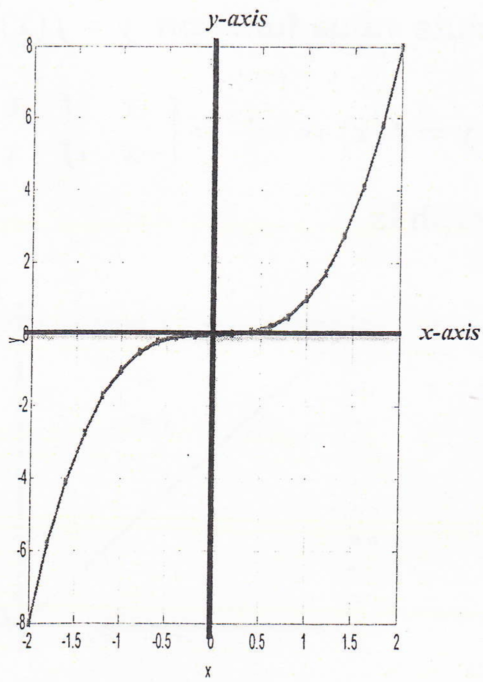
3) A function $y = f(x) = x^r$ where r is a real number is called a power function .

Example 2.3.2 :

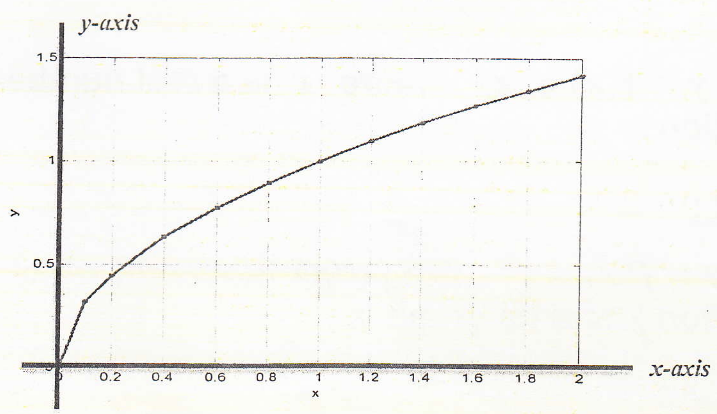
The function $y = f(x) = x^2$ is a power function (which is also a quadratic function) and its graph is



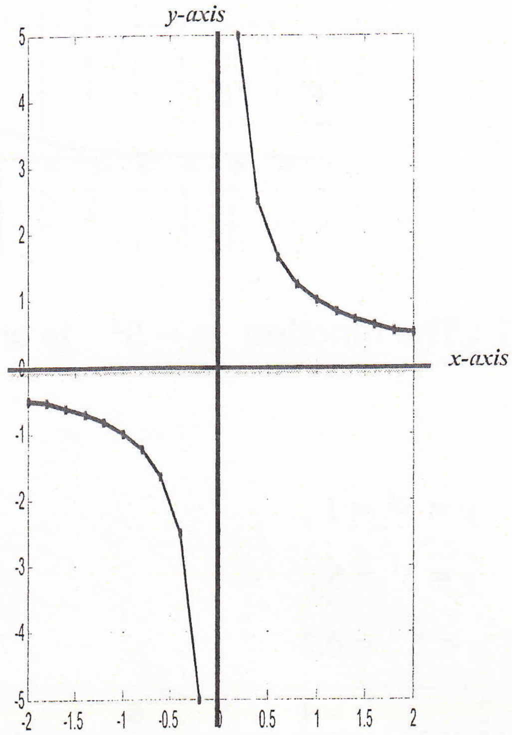
Example 2.3.3 : The function $y = f(x) = x^3$ is a power function and its graph is



Example 2.3.4 : The function $y = f(x) = \sqrt{x}$ is a power function and its graph is



Example 2.3.5 : The function $y = f(x) = \frac{1}{x}$ is a power function and its graph is



4) Let a be a positive real number other than 1 . The function $y = f(x) = a^x$ is called the exponential function with base a .

Example 2.3.6 : Graph the exponential function $y = 2^x$

Answer : To draw the graph of $y = 2^x$, we can make use of a table give values for x and find the corresponding values for y

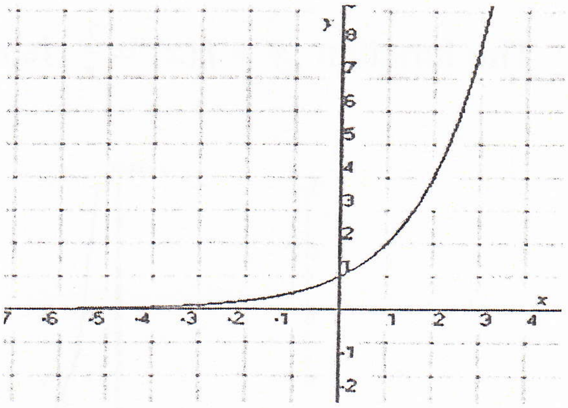
$x = 0$ gives $y = 2^0 = 1$,

$x = 1$ gives $y = 2^1 = 2$,

$x = -1$ gives $y = 2^{-1} = \frac{1}{2}$.

Following the process we make the table

x	−4	−3	−2	−1	0	1	2	3	4
2^x	0.0625	0.125	0.25	0.5	1	2	4	8	16



Example 2.3.7 : The function $y = 5^x$ is an exponential function and its graph is

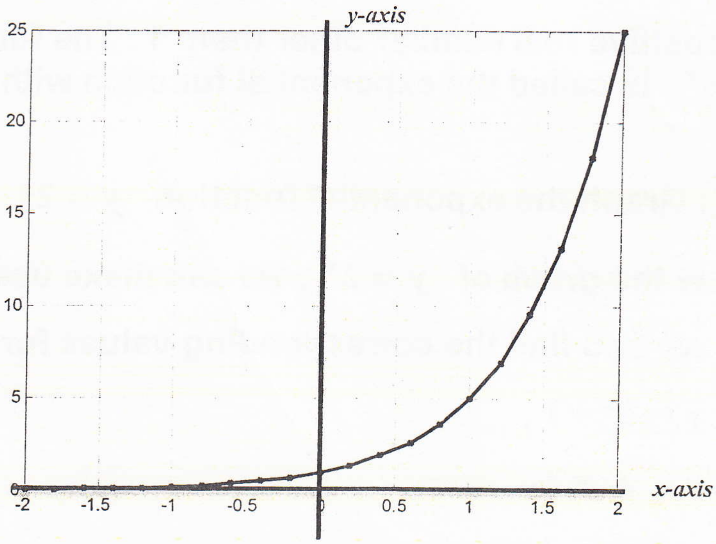
Answer :

$x = 0$ gives $y = 5^0 = 1$,

$x = 1$ gives $y = 5^1 = 5$,

$x = -1$ gives $y = 5^{-1} = 0.2$

x	-2	-1	0	1	2
5^x	0.04	0.2	1	5	25



Exercise 2.3.8 : Graph the exponential function $y = 10^x$.

The properties of exponential function and their graph

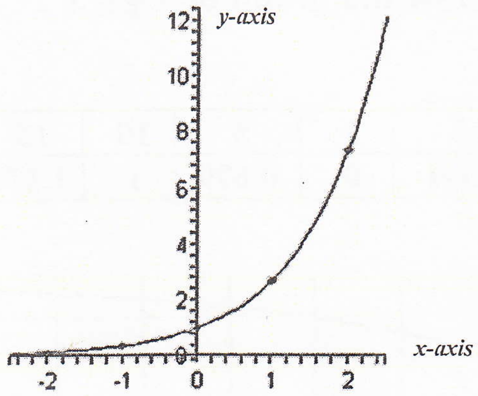
- The domain is R (set of real numbers) .
- The range is R^+ (set of positive real numbers) .
- The graph is always continuous (no break in the graph) .

Rules of Exponents : If $a > 0$ and $b > 0$, the following rules of exponent should be hold for all real numbers x and y :

1. $a^x \times a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $a^0 = 1$
4. $\dfrac{1}{a^x} = a^{-x}$
5. $(a^x)^y = (a^y)^x = a^{xy}$
6. $(ab)^x = a^x b^x$
7. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

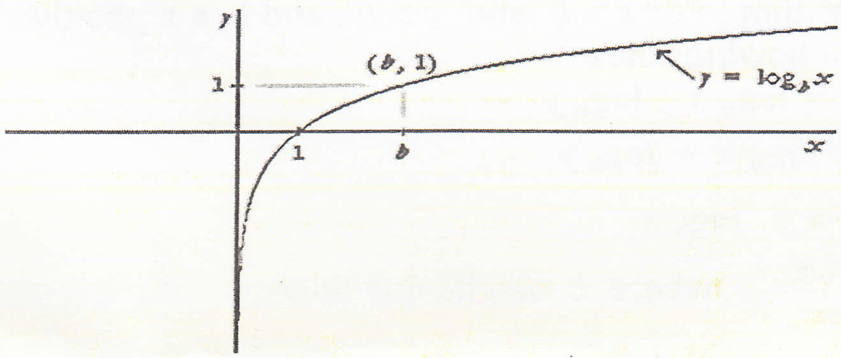
5) The function $y = e^x$ is called the natural exponential function whose base is $e \cong 2.718281828$, and its graph is

x	-2	-1	0	1	2
e^x	0.1353	0.3679	1	2.718	7.389



Remark : Graph of e^x and e^{-x} are reflections of each other .

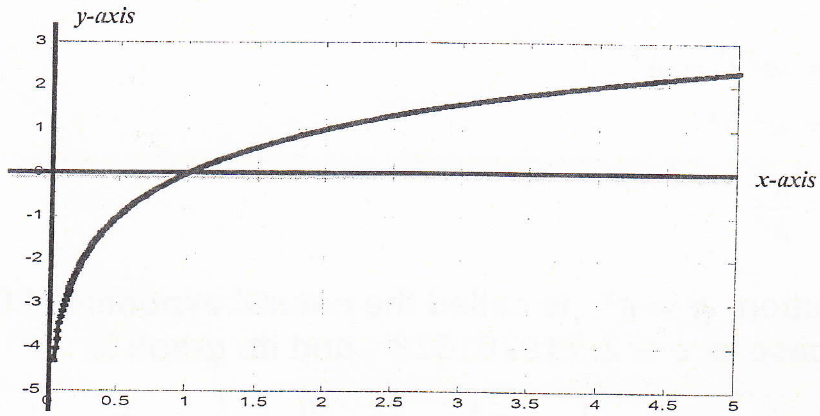
6) The function $y = \log_b x$ is called the logarithm function with base b where b is a positive number $\neq 1$; and $x > 0$, and the graph of $y = \log_b x$ where b is greater than 1 is the following graph



Remark : $y = \log_b x$ means that $x = b^y$.

Example 2.3.9 : The function $y = \log_2 x$ is a logarithm function with base 2 and its graph is

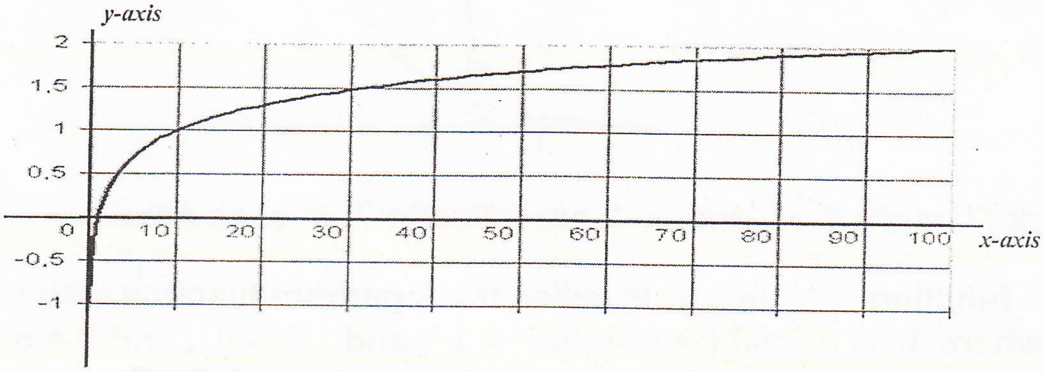
x	0.25	0.5	1	2	4
$y = \log_2 x$	-2	-1	0	1	2



Example 2.3.10 : Draw the graph of $\log_{10} x$.

Answer :

x	0.5	1	5	10	15	20	50	100
$y = \log_{10} x$	-0.301	0	0.699	1	1.176	1.301	1.699	2



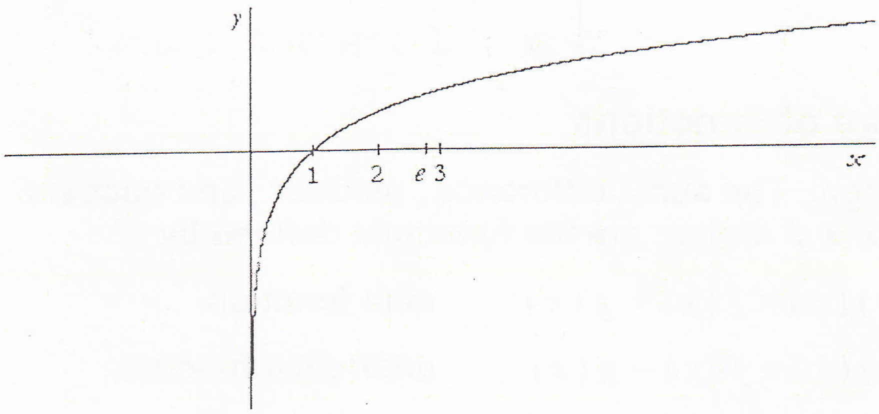
Rules of logarithm : For $x > 0$ and $y > 0$, and b is a positive number $\neq 1$ we have the following rules :

1. $\log_b xy = \log_b x + \log_b y$
2. $\log_b \frac{x}{y} = \log_b x - \log_b y$
3. $\log_b x^y = y \, . \, \log_b x$
4. $\log_b a = \dfrac{\log_c a}{\log_c b}$, where c can be any base .

Remarks :

- The logarithm of any number to the base of the same number will be 1 ($\log_b b = 1$, $\log_5 5 = 1$ etc ...) .
- Logarithm of 1 to any base is 0 ($\log_b 1 = 0$, $\log_3 1 = 0$ etc ...) .
- The logarithm function is defined only for positive numbers .
- The domain of the logarithm function is R^+ .
- The range of the logarithm function is R .

7) The logarithm function with base e is called the natural logarithm function and will be denoted by $y = \ln x$ (i.e. $y = \log_e x = \ln x$) and its graph is



Remarks :

- $\ln e = 1$ (since $\ln e = \log_e e$)
- $\ln 1 = 0$

Exercise 2.3.12 : Draw the graph for the following logarithmic functions:

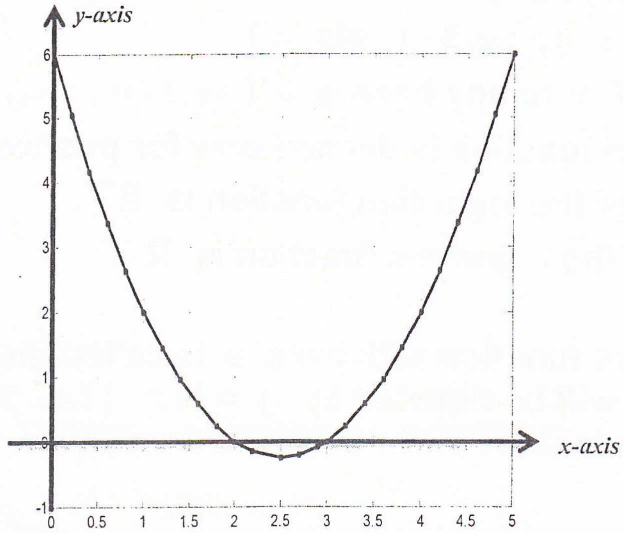
1. $\log_5 x$
2. $\log_8 x$
3. $\log_3 x$

8) A polynomial function is defined as

$y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where

a_0 , a_1 , ... , a_{n-1} , a_n are constants .

Example 2.3.13 : The function $y = x^2 - 5x + 6$ is a polynomial function .



Algebra of Functions

Definition: The sum , difference , product , and quotient of the functions f and g are the functions defined by

$(f+g)(x) = f(x) + g(x)$ sum function

$(f-g)(x) = f(x) - g(x)$ difference function

$(f.g)(x) = f(x).g(x)$ product function

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $g(x) \neq 0$ quotient function

The domain of each function is the intersection of the domains of f and g , with the exception that the values of x where $g(x) = 0$ must be excluded from the domain of the quotient function .

Definition: Let f and g be functions , then $f \circ g$ is called the composite of g and f and is defined by the equation $(f \circ g)(x) = f(g(x))$.

The domain of $f \circ g$ is the set
$\mathbf{D} = \{ x \in \text{domain } g : g(x) \in \text{domain } f \}$.

Example 2.3.14 : Let f and g be the functions defined by $f(x) = x - 7$ and $g(x) = x^2 + 5$. Find the functions $f+g$, $f-g$, $f.g$, $\frac{g}{f}$, $f \circ g$, $g \circ f$ and find their domains .

Solution :

$(f+g)(x)=f(x)+g(x) = x-7 + x^2+5 = x^2+x-2$

$(f-g)(x)=f(x)-g(x) = x-7 - x^2-5 = -x^2+x-12$

$(f.g)(x)=f(x).g(x) = (x-7).(x^2+5) = x^3-7x^2+5x-35$

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{x^2+5}{x-7}$$

$(f\circ g)(x)=f(g(x)) = f(x^2+5) = x^2+5-7 = x^2-2$

$(g\circ f)(x)= g(f(x))= g(x-7) = (x-7)^2+5$

$= x^2-14x+49+5 = x^2-14x+54$

The domain of f = R

The domain of g = R

The intersection of the domains of f and g is R

Thus the domain of each of the functions $f+g$, $f-g$, $f.g$, $f\circ g$, and $g\circ f$ is R .

The domain of $\frac{g}{f}$ = R$-\{7\}$.

Remark : The domain of any polynomial function is R .

Example 2.3.15 : Let f and g be the functions defined by $f(x)=x+5$ and $g(x)=x^2-3$, Find $f\circ g(x)$, $g\circ f(x)$, $f\circ g(3)$ and $g\circ f(3)$.

Solution: $f\circ g(x) = f(g(x)) = f(x^2-3)$

$= x^2-3+5$

$= x^2+2$

$g\circ f(x) = g(f(x)) = g(x+5)$

$= (x+5)^2-3$

$= x^2+10x+25-3$

$= x^2+10x+22$

$f \circ g(3) = (3)^2 + 2 = 9 + 2 = 11$

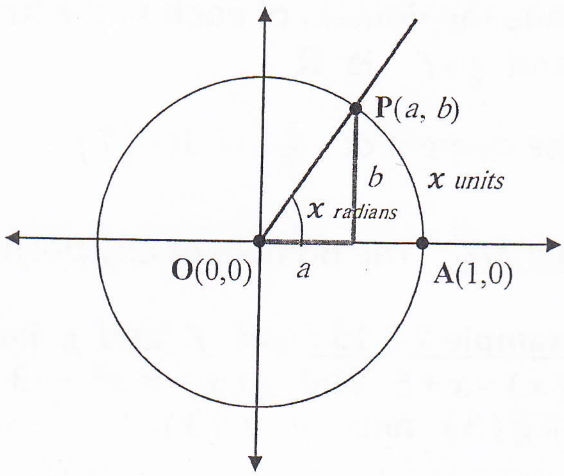
$g \circ f(3) = (3)^2 + 10\,(3) + 22 = 9 + 30 + 22 = 61$

Exersice 2.3.16 : Let f and g be the functions defined by $f(x) = x - 4$ and $g(x) = \sqrt{x}$. Find the functions $f + g$, $f - g$, $f \cdot g$, $\frac{f}{g}$ and find their domains .

S 2.4 : Unit Circle and Basic Trigonometric Functions

Definition 1: Let x be any real number and let U be the unit circle with equation $a^2 + b^2 = 1$ (the centre of the circle U is the point O (0,0) , and the radius of the circle U equals 1) . Start from the point A(1,0) on U and proceed counterclockwise if x is positive and clockwise if x is negative around the unit circle U until an arc length of $|x|$ has been covered . Let P(a, b) be the point at the terminal end of the arc . The measurement of the angle AOP is x radians .

If x radians = t° (degrees) , then the following six trigonometric functions of x are defined in terms of the coordinates of the circular point P(a, b) :



1) $y = \sin x = b = \sin\,(x \text{ radians}) = \sin\,(t \text{ degrees}) = \sin t^\circ$

2) $y = \cos x = a = \cos\,(x \text{ radians}) = \cos\,(t \text{ degrees}) = \cos t^\circ$

3) $y = \tan x = \frac{b}{a} \qquad (a \neq 0)$

$= \tan\,(x \text{ radians}) = \tan\,(t \text{ degrees}) = \tan t^\circ$

4) $y = \cot x = \frac{a}{b} \qquad (b \neq 0)$

$= \cot\,(x \text{ radians}) = \cot\,(t \text{ degrees}) = \cot t^\circ$

5) $y = \sec x = \frac{1}{a}$ $\quad (a \neq 0)$

$= \sec(x \text{ radians}) = \sec(t \text{ degrees}) = \sec t^\circ$

6) $y = \csc x = \frac{1}{b}$ $\quad (b \neq 0)$

$= \csc(x \text{ radians}) = \csc(t \text{ degrees}) = \csc t^\circ$

Remark 1: Definition 1 uses the standard function notation , $y = f(x)$, with f replaced by the name of a particular trigonometric function . For example , $y = \cos x$ actually means $y = \cos(x)$ and $\cos t^\circ$ actually means $\cos(t^\circ)$.

Remark 2: Remember that $t^\circ = t \times \frac{\pi}{180}$ *radians* and $x \text{ radians} = (x \times \frac{180}{\pi})^\circ$.

Theorem 1:

For any real number x we have the following trigonometric identities :

1) $\csc x = \frac{1}{\sin x}$.

2) $\sec x = \frac{1}{\cos x}$.

3) $\cot x = \frac{1}{\tan x}$.

4) $\tan x = \frac{\sin x}{\cos x}$.

5) $\cot x = \frac{\cos x}{\sin x}$.

6) $\sin(-x) = -\sin(x)$.

7) $\cos(-x) = \cos(x)$.

8) $\tan(-x) = -\tan(x)$.

9) $\cot(-x) = -\cot(x)$.

10) $\sin^2 x + \cos^2 x = 1$.

11) $\sec^2 x = \tan^2 x + 1$.

12) $\csc^2 x = \cot^2 x + 1$.

S 2.5: Graphs of Sine and Cosine Functions

2.5.1: Table for values of $\sin x$, $\cos x$, and $\tan x$ for selected values of x

Values of x	Degrees	0	30	45	60	90
	Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin x$		0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$		1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan x$		0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

Values of x	Degrees	120	135	150	180	270
	Radians	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$
$\sin x$		$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
$\cos x$		$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0
$\tan x$		$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	Undefined

Definition: A function f is periodic if there exists a positive real number p such that $f(x)=f(x+p)$ for all x in the domain of f.
The smallest such positive number p is the period of f.

Remarks :

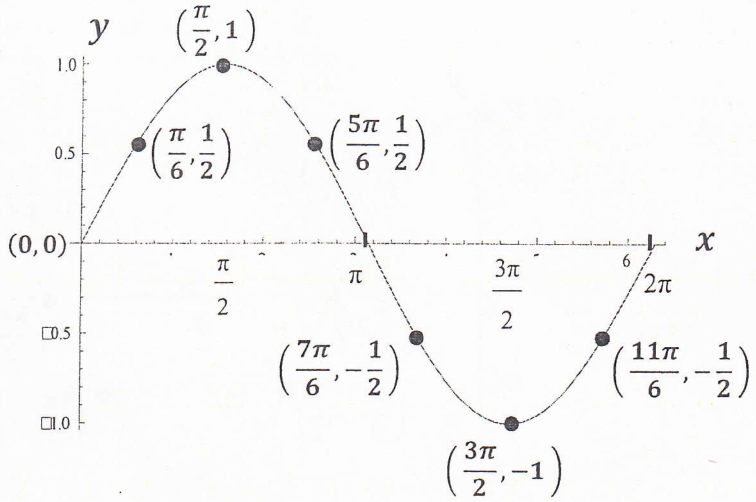
1) The functions $\sin x$, $\cos x$, $\sec x$, and $\csc x$ are periodic functions with period 2π.

2) The functions $\tan x$ and $\cot x$ are periodic functions with period π.

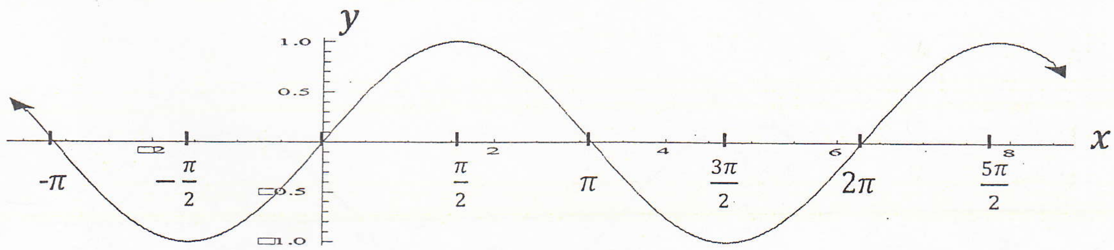
2.5.2: The Graph of $\sin x$

The graph of the function $y = \sin x$ is the line passing through all the points $(x, \sin x)$ on the xy-plane .

The graph of the function $y = \sin x$ for the interval $[0, 2\pi]$ is the line passing through the points $(0, 0)$, $(\frac{\pi}{6}, \frac{1}{2})$, $(\frac{\pi}{2}, 1)$, $(\frac{5\pi}{6}, \frac{1}{2})$, $(\pi, 0)$, $(\frac{7\pi}{6}, -\frac{1}{2})$, $(\frac{3\pi}{2}, -1)$, $(\frac{11\pi}{6}, -\frac{1}{2})$, and $(2\pi, 0)$ which is shown in the following figure



The graph of the function $y = \sin x$ is shown in the following figure



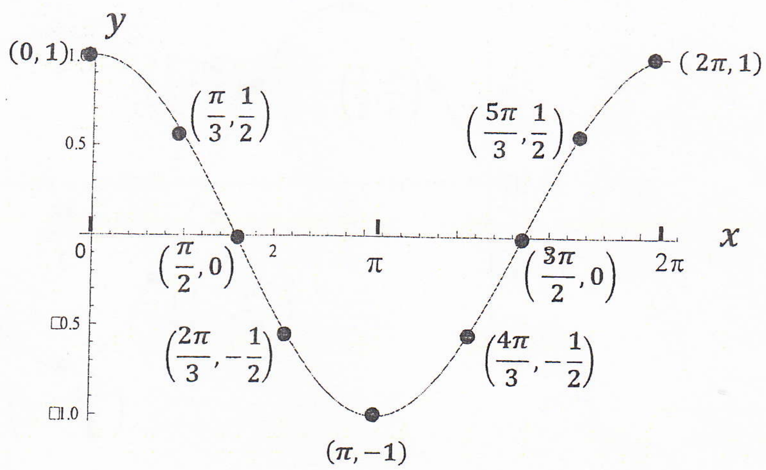
The period of the function $y = \sin x$ is 2π . The domain of the function $y = \sin x$ is the set of all real numbers R .

The range of the function $y = \sin x$ is the interval $[-1, 1]$.

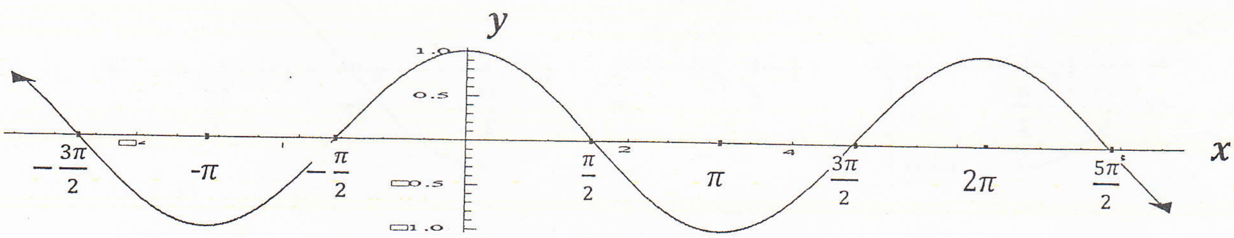
2.5.3: The Graph of $\cos x$

The graph of the function $y = \cos x$ is the line passing through all the points $(x, \cos x)$ on the xy-plane .

The graph of the function $y = \cos x$ for the interval $[0, 2\pi]$ is the line Passing through the points $(0, 1)$, $(\frac{\pi}{3}, \frac{1}{2})$, $(\frac{\pi}{2}, 0)$, $(\frac{2\pi}{3}, -\frac{1}{2})$, $(\pi, -1)$, $(\frac{4\pi}{3}, -\frac{1}{2})$, $(\frac{3\pi}{2}, 0)$, $(\frac{5\pi}{3}, \frac{1}{2})$, and $(2\pi, 1)$ which is shown in the following figure



The graph of the function $y = \cos x$ is shown in the following figure

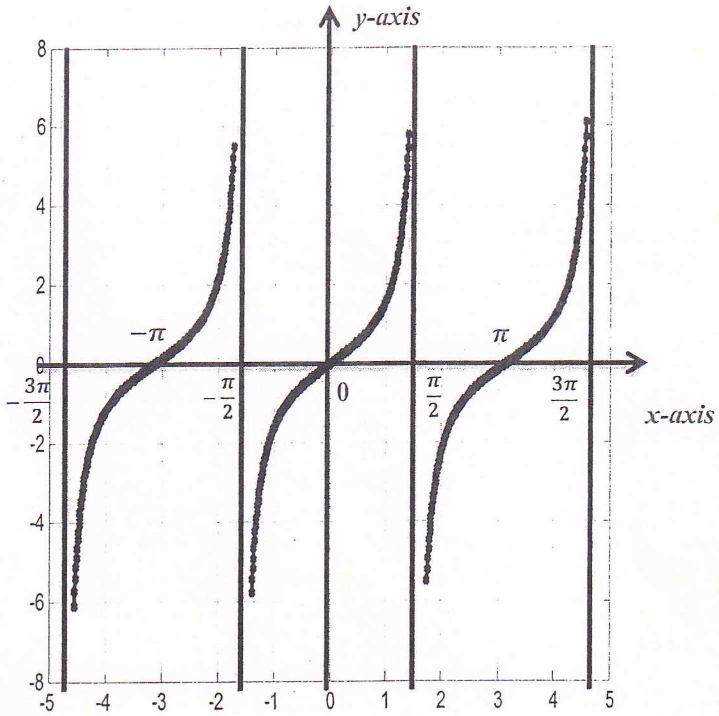


The period of the function $y = \cos x$ is 2π .
The domain of the function $y = \cos x$ is the set of all real numbers R .
The range of the function $y = \cos x$ is the interval $[-1, 1]$.

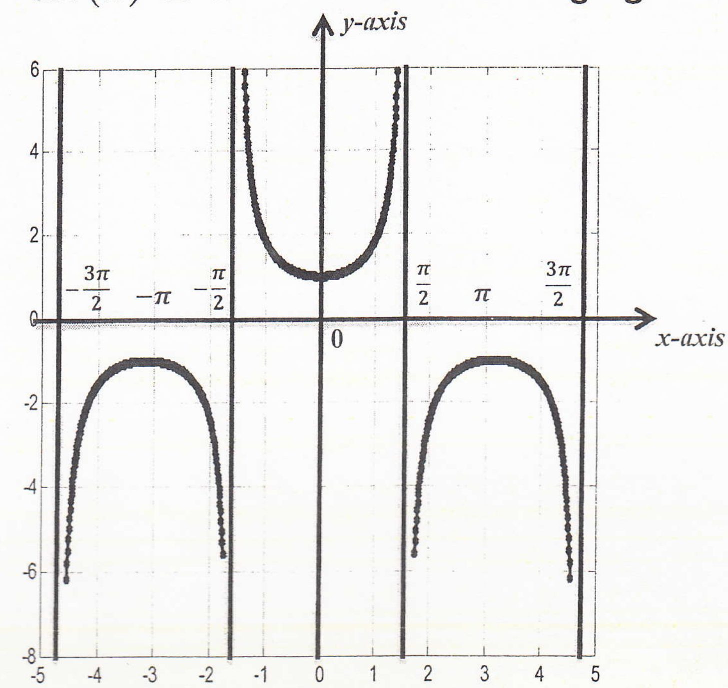
2.5.4: The Graphs of $\tan x$ and $\sec x$

The graph of the function $y = \tan(x)$ is the line passing through all the points $(x, \tan x)$ on the xy-plane .

The graph of $y = \tan(x)$ is shown in the following figure



The graph of $y = \sec(x)$ is shown in the following figure



Exercise: Draw the graph of the following trigonometric functions :

1) $y = \csc(x)$

2) $y = \cot(x)$