رياضيات الحاسبات
المرحلة الأولى

مكتب قطر الندى
للطباعة والاستنساخ
مجاور الجامعة المستنصرية
عمل وطباعة بحوث والتقارير
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CH2: Functions

S2.1: Functions and Their Graphs

Definition: A function \( f \) (or a mapping) from a set \( A \) to a set \( B \) is a rule that assigns to each element \( a \) of \( A \) exactly one element \( b \) of \( B \). The set \( A \) is called the domain of \( f \) and the set \( B \) is called the codomain of \( f \). If \( f \) assigns \( b \) to \( a \), then \( b \) is called the image of \( a \) under \( f \). The subset of \( B \) comprised of all the images of elements of \( A \) under \( f \) (which is denoted by \( f(A) \)) is called the image of \( A \) under \( f \) (or the range of \( f \)).

We use \( f: A \rightarrow B \) to mean that \( f \) is a function from \( A \) to \( B \). We will write \( f(a) = b \) to indicate that \( b \) is the image of \( a \) under \( f \).

Example 2.1.1:
Let \( A = \{2, 4, 5\} \), \( B = \{1, 2, 3, 6\} \), and \( f: A \rightarrow B \) be the function defined by \( f(2) = 1 \), \( f(4) = 3 \), \( f(5) = 6 \). Then the domain of \( f \) is \( A = \{2, 4, 5\} \), the codomain of \( f \) is \( B = \{1, 2, 3, 6\} \), and the range of \( f \) is \( \{1, 3, 6\} \).

Counter example:
Let \( C = \{1, 2, 3, 4\} \) and \( D = \{2, 3, 4, 5\} \), and let \( h \) be the rule defined by \( h(1) = 2 \), \( h(1) = 4 \), \( h(2) = 3 \), \( h(3) = 5 \), \( h(4) = 4 \), then \( h \) is not a function from \( C \) to \( D \) since there are two different elements \( 2 \) and \( 4 \) belong to \( D \) are assigned to the same element \( 2 \) of \( C \).

Example 2.1.2: Find the domain and the range of the function \( f \) defined by \( f(x) = \sqrt{x+10} \).

Solution: For \( y = f(x) = \sqrt{x+10} \) to be real, \( x + 10 \) must be greater than or equal to 0. That is, \( x + 10 \geq 0 \) which means that \( x \geq -10 \). Thus the domain is \( \{x: x \geq -10\} \) and the range is \( \{y: y \geq 0\} \).

Exercises:

1) Let \( A = \{2, 4, 5, 7\} \), \( B = \{1, 2, 3, 6, 9\} \), and \( f: A \rightarrow B \) be the function defined by \( f(2) = 9 \), \( f(4) = 3 \), \( f(5) = 6 \), \( f(7) = 2 \). Find the domain of \( f \), the codomain of \( f \), and the range of \( f \).
2) Let \( f \) be a function defined by \( f(x) = \frac{1}{x+2} \). Find the domain and the range of the function \( f \).

3) Find the domain and the range of the function \( f \) defined by 
\[
f(x) = \sqrt{2x-9}.
\]

**Definition:** The graph of a function \( f \) is the line passing through all the points \((x, f(x))\) on the \( xy \)-plane.

**Definition:** The \( y \)-coordinate of the point where a graph of a function \( f \) intersect the \( y \)-axis is called the \( y \)-intercept of the function.

**Definition:** The \( x \)-coordinate of a point where a graph of a function \( f \) intersects the \( x \)-axis is called an \( x \)-intercept of the function.

**Remarks:**

1) The graph of any function \( f \) has at most one \( y \)-intercept. The graph of the function \( f \) has exactly one \( y \)-intercept if 0 is in the domain of the function \( f \) and the \( y \)-intercept is \( f(0) \).

2) The graph of any function \( f \) has no \( x \)-intercept if there is no \( x \) in the domain of the function \( f \) such that \( f(x) = 0 \). The graph of a function \( f \) has one or more than one \( x \)-intercepts if \( f(x) = 0 \) for some \( x \) in the domain of \( f \), and the number of \( x \)-intercepts is the number of the distinct solutions of the equation \( f(x) = 0 \).

**Properties of Functions:**

1) A function \( y = f(x) \) is called an even function of \( x \) if \( f(-x) = f(x) \), \( \forall \ x \).

2) A function \( y = f(x) \) is called an odd function of \( x \) if \( f(-x) = -f(x) \), \( \forall \ x \).

**S2.2 : Linear Functions and their Graphs**

**Definition:** A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is called a linear function if \( f \) is defined by \( f(x) = ax + b \), \( a \neq 0 \) where \( a \) and \( b \) are real numbers.
Example 2.2.1: The function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = 3x + 12 \) is a linear function.

Example 2.2.2: The function \( g : \mathbb{R} \to \mathbb{R} \) defined by \( g(x) = x - 0.2 \) is a linear function.

Example 2.2.3: The function \( h : \mathbb{R} \to \mathbb{R} \) defined by \( h(x) = -\frac{3}{2}x + 1 \) is a linear function.

Example 2.2.4: Let \( f : \mathbb{R} \to \mathbb{R} \) be the linear function defined by \( f(x) = 4x + 10 \). Find the \( x \)-intercept and the \( y \)-intercept of \( f \).

**Solution:**

\[
\begin{align*}
f(x) &= 0 \\
4x + 10 &= 0 \\
4x &= -10 \\
x &= \frac{-10}{4} = -2.5
\end{align*}
\]

Therefore the \( x \)-intercept is \(-2.5\).

\( f(0) = 10 \Rightarrow \) the \( y \)-intercept is \(10\).

Example 2.2.5: Let \( g : \mathbb{R} \to \mathbb{R} \) be the linear function defined by \( g(x) = \frac{1}{5}x - 6 \). Find the \( x \)-intercept and the \( y \)-intercept of \( g \).

**Solution:**

\[
\begin{align*}
g(x) &= 0 \\
\frac{1}{5}x - 6 &= 0 \\
\frac{1}{5}x &= 6 \\
x &= 30
\end{align*}
\]

Therefore the \( x \)-intercept is \(30\).

\( g(0) = -6 \Rightarrow \) the \( y \)-intercept is \(-6\).

Graph of a linear function:

The graph of a linear function \( f \) is the straight line passing through the two points \((a, 0)\) and \((0, b)\) where \(a\) is the \( x \)-intercept of the function \( f \) and \(b\) is the \( y \)-intercept of the function \( f \).

Remark: The graph of any linear function \( f \) has exactly one \( x \)-intercept and has exactly one \( y \)-intercept.
Example 2.2.6: Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be the linear function defined by 
\[ f(x) = -2x + 7 \]. Find the \( x \)-intercept and the \( y \)-intercept of \( f \),
then graph the function \( f \).

**Solution:** 
\[ f(x) = 0 \Rightarrow -2x + 7 = 0 \]
\[ \Rightarrow -2x = -7 \]
\[ \Rightarrow x = \frac{-7}{-2} = 3.5 \]

Therefore the \( x \)-intercept is 3.5.

\[ f(0) = 7 \Rightarrow \text{the } y \text{-intercept is 7.} \]

Thus the graph of the function \( f \) is the straight line passing through the two points \((3.5, 0)\) and \((0, 7)\).

Thus the graph of the function \( f \) is the following graph:

![Graph of f(x) = -2x + 7](image)

Example 2.2.7: Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be the linear function defined by
\[ g(x) = 4x + 12 \]. Find the \( x \)-intercept and the \( y \)-intercept of \( g \), then
graph the function \( g \).

**Solution:** 
\[ g(x) = 0 \Rightarrow 4x + 12 = 0 \]
\[ \Rightarrow 4x = -12 \]
\[ \Rightarrow x = \frac{-12}{4} = -3 \]

Therefore the \( x \)-intercept is \(-3\).

\[ g(0) = 12 \Rightarrow \text{the } y \text{-intercept is 12.} \]
Thus the graph of the function \( g \) is the straight line passing through the two points \((-3,0)\) and \((0,12)\).

Thus the graph of the function \( g \) is the following graph

\[
g(x) = 4x + 12
\]

Exercises:

1) Let \( f: \mathbb{R} \to \mathbb{R} \) be the linear function defined by \( f(x) = 3x - 10 \).
   Find the \( x \)-intercept and the \( y \)-intercept of \( f \).

2) Let \( g: \mathbb{R} \to \mathbb{R} \) be the linear function defined by \( g(x) = 0.3x + 0.7 \).
   Find the \( x \)-intercept and the \( y \)-intercept of \( g \).

3) Let \( f: \mathbb{R} \to \mathbb{R} \) be the linear function defined by \( f(x) = -4x + 8 \).
   Find the \( x \)-intercept and the \( y \)-intercept of \( f \), then graph the function \( f \).

4) Let \( g: \mathbb{R} \to \mathbb{R} \) be the linear function defined by \( g(x) = 5x + 15 \).
   Find the \( x \)-intercept and the \( y \)-intercept of \( g \), then graph the function \( g \).

S2.3 : Some well-known Functions and their Graphs

1) A function \( f(x) = c \) where \( c \) is a fixed number is called a constant function.
Example 2.3.1: The function \( y = f(x) = 1 \) is a constant function and its graph is

\[
\begin{align*}
\text{Example 2.3.2:} & \\
\text{The function } y = f(x) = x^2 & \text{ is a power function (which is also a quadratic function) and its graph is}
\end{align*}
\]
Example 2.3.3: The function \( y = f(x) = x^2 \) is a power function and its graph is

Example 2.3.4: The function \( y = f(x) = \sqrt{x} \) is a power function and its graph is
Example 2.3.5: The function $y = f(x) = \frac{1}{x}$ is a power function and its graph is

4) Let $a$ be a positive real number other than 1. The function $y = f(x) = a^x$ is called the exponential function with base $a$.

Example 2.3.6: Graph the exponential function $y = 2^x$

Answer: To draw the graph of $y = 2^x$, we can make use of a table that gives values for $x$ and finds the corresponding values for $y$

$x = 0$ gives $y = 2^0 = 1$,

$x = 1$ gives $y = 2^1 = 2$,

$x = -1$ gives $y = 2^{-1} = \frac{1}{2}$.

Following the process we make the table

<table>
<thead>
<tr>
<th>$x$</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^x$</td>
<td>0.0625</td>
<td>0.125</td>
<td>0.25</td>
<td>0.5</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
</tr>
</tbody>
</table>
Example 2.3.7: The function \( y = 5^x \) is an exponential function and its graph is

\[ \begin{array}{c|ccccc}
 x & -2 & -1 & 0 & 1 & 2 \\
 5^x & 0.04 & 0.2 & 1 & 5 & 25 \\
\end{array} \]

Exercise 2.3.8: Graph the exponential function \( y = 10^x \).

The properties of exponential function and their graph
- The domain is \( \mathbb{R} \) (set of real numbers).
- The range is \( \mathbb{R}^+ \) (set of positive real numbers).
- The graph is always continuous (no break in the graph).
Rules of Exponents: If \( a > 0 \) and \( b > 0 \), the following rules of exponent should be hold for all real numbers \( x \) and \( y \):

1. \( a^x \times a^y = a^{x+y} \)
2. \( \frac{a^x}{a^y} = a^{x-y} \)
3. \( a^0 = 1 \)
4. \( \frac{1}{a^x} = a^{-x} \)
5. \( (a^x)^y = (a^y)^x = a^{x\cdot y} \)
6. \( (ab)^x = a^x b^x \)
7. \( \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \)

5) The function \( y = e^x \) is called the natural exponential function whose base is \( e \approx 2.718281828 \), and its graph is

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^x )</td>
<td>(0.1353)</td>
<td>(0.3679)</td>
<td>(1)</td>
<td>(2.718)</td>
<td>(7.389)</td>
</tr>
</tbody>
</table>

Remark: Graph of \( e^x \) and \( e^{-x} \) are reflections of each other.

6) The function \( y = \log_b x \) is called the logarithm function with base \( b \) where \( b \) is a positive number \( \neq 1 \); and \( x > 0 \), and the graph of \( y = \log_b x \) where \( b \) is greater than \( 1 \) is the following graph
Remark: \( y = \log_b x \) means that \( x = b^y \).

**Example 2.3.9:** The function \( y = \log_2 x \) is a logarithm function with base 2 and its graph is

\[
\begin{array}{c|c|c|c|c|c}
 x & 0.25 & 0.5 & 1 & 2 & 4 \\
 y = \log_2 x & -2 & -1 & 0 & 1 & 2 \\
\end{array}
\]

![Graph of \( y = \log_2 x \)](image)

**Example 2.3.10:** Draw the graph of \( \log_{10} x \).

**Answer:**

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
 x & 0.5 & 1 & 5 & 10 & 15 & 20 & 50 & 100 \\
 y = \log_{10} x & -0.301 & 0 & 0.699 & 1 & 1.176 & 1.301 & 1.699 & 2 \\
\end{array}
\]

![Graph of \( y = \log_{10} x \)](image)

**Rules of logarithm:** For \( x > 0 \) and \( y > 0 \), and \( b \) is a positive number \( \neq 1 \) we have the following rules:

1. \( \log_b x y = \log_b x + \log_b y \)
2. \( \log_b \frac{x}{y} = \log_b x - \log_b y \)
3. \( \log_b x^y = y \cdot \log_b x \)
4. \( \log_b a = \frac{\log_c a}{\log_c b} \), where \( c \) can be any base.
Remarks:

- The logarithm of any number to the base of the same number will be 1 (log₆₆₆ = 1, log₅₅₅ = 1, etc...).
- Logarithm of 1 to any base is 0 (log₆₁ = 0, log₅₁ = 0, etc...).
- The logarithm function is defined only for positive numbers.
- The domain of the logarithm function is R⁺.
- The range of the logarithm function is R.

7) The logarithm function with base e is called the natural logarithm function and will be denoted by \( y = \ln x \) (i.e. \( y = \log_e x = \ln x \)) and its graph is

![Graph of \( y = \ln x \)]

Remarks:

- \( \ln e = 1 \) (since \( \ln e = \log_e e \))
- \( \ln 1 = 0 \)

Exercise 2.3.12: Draw the graph for the following logarithmic functions:

1. \( \log_5 x \)
2. \( \log_8 x \)
3. \( \log_3 x \)

8) A polynomial function is defined as

\[
y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad \text{where}
\]

\[
a_0, a_1, \ldots, a_{n-1}, a_n \quad \text{are constants}.
\]
Example 2.3.13: The function \( y = x^2 - 5x + 6 \) is a polynomial function.

**Algebra of Functions**

**Definition:** The sum, difference, product, and quotient of the functions \( f \) and \( g \) are the functions defined by

\[
(f + g)(x) = f(x) + g(x) \quad \text{sum function}
\]
\[
(f - g)(x) = f(x) - g(x) \quad \text{difference function}
\]
\[
(f \cdot g)(x) = f(x) \cdot g(x) \quad \text{product function}
\]
\[
\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \quad g(x) \neq 0 \quad \text{quotient function}
\]

The domain of each function is the intersection of the domains of \( f \) and \( g \), with the exception that the values of \( x \) where \( g(x) = 0 \) must be excluded from the domain of the quotient function.

**Definition:** Let \( f \) and \( g \) be functions, then \( f \circ g \) is called the composite of \( g \) and \( f \) and is defined by the equation

\[
(f \circ g)(x) = f(g(x))
\]

The domain of \( f \circ g \) is the set

\[ D = \{ x \in \text{domain } g : g(x) \in \text{domain } f \} \]

**Example 2.3.14:** Let \( f \) and \( g \) be the functions defined by

\( f(x) = x - 7 \) and \( g(x) = x^2 + 5 \). Find the functions \( f + g \), \( f - g \), \( f \cdot g \), \( \frac{g}{f} \), \( f \circ g \), \( g \circ f \) and find their domains.
Solution:

\(( f + g )(x) = f(x) + g(x) = x - 7 + x^2 + 5 = x^2 + x - 2 \)

\(( f - g )(x) = f(x) - g(x) = x - 7 - x^2 - 5 = -x^2 + x - 12 \)

\(( f \cdot g )(x) = f(x) \cdot g(x) = (x - 7) \cdot (x^2 + 5) = x^3 - 7x^2 + 5x - 35 \)

\( \left( \frac{g}{f} \right)(x) = \frac{g(x)}{f(x)} = \frac{x^2 + 5}{x - 7} \)

\(( f \circ g )(x) = f(g(x)) = f(x^2 + 5) = x^2 + 5 - 7 = x^2 - 2 \)

\(( g \circ f )(x) = g(f(x)) = g(x - 7) = (x - 7)^2 + 5 = x^2 - 14x + 49 + 5 = x^2 - 14x + 54 \)

The domain of \( f = \mathbb{R} \)

The domain of \( g = \mathbb{R} \)

The intersection of the domains of \( f \) and \( g \) is \( \mathbb{R} \)

Thus the domain of each of the functions \( f + g \), \( f - g \), \( f \cdot g \), \( f \circ g \), and \( g \circ f \) is \( \mathbb{R} \).

The domain of \( \frac{g}{f} = \mathbb{R} - \{7\} \).

Remark: The domain of any polynomial function is \( \mathbb{R} \).

Example 2.3.15: Let \( f \) and \( g \) be the functions defined by \( f(x) = x + 5 \) and \( g(x) = x^2 - 3 \). Find \( f \circ g(x) \), \( g \circ f(x) \), \( f \circ g(3) \) and \( g \circ f(3) \).

Solution: \( f \circ g(x) = f(g(x)) = f(x^2 - 3) = x^2 - 3 + 5 = x^2 + 2 \)

\( g \circ f(x) = g(f(x)) = g(x + 5) = (x + 5)^2 - 3 = x^2 + 10x + 25 - 3 = x^2 + 10x + 22 \)
\[ f \circ g(3) = (3)^2 + 2 = 9 + 2 = 11 \]
\[ g \circ f(3) = (3)^2 + 10(3) + 22 = 9 + 30 + 22 = 61 \]

Exercise 2.3.16: Let \( f \) and \( g \) be the functions defined by
\[ f(x) = x - 4 \quad \text{and} \quad g(x) = \sqrt{x} \] . Find the functions \( f + g \), \( f - g \), \( f \cdot g \) and \( \frac{f}{g} \) and find their domains.

2.4: Unit Circle and Basic Trigonometric Functions

Definition 1: Let \( x \) be any real number and let \( U \) be the unit circle with equation \( a^2 + b^2 = 1 \) (the centre of the circle \( U \) is the point \( O(0,0) \), and the radius of the circle \( U \) equals 1). Start from the point \( A(1,0) \) on \( U \) and proceed counterclockwise if \( x \) is positive and clockwise if \( x \) is negative around the unit circle \( U \) until an arc length of \( |x| \) has been covered. Let \( P(a,b) \) be the point at the terminal end of the arc. The measurement of the angle \( \angle AOP \) is \( x \) radians.

If \( x \) radians = \( t^\circ \) (degrees), then the following six trigonometric functions of \( x \) are defined in terms of the coordinates of the circular point \( P(a,b) \):

1) \[ y = \sin x = b = \sin (x \text{ radians}) = \sin (t \text{ degrees}) = \sin t^\circ \]
2) \[ y = \cos x = a = \cos (x \text{ radians}) = \cos (t \text{ degrees}) = \cos t^\circ \]
3) \[ y = \tan x = \frac{b}{a} \quad (a \neq 0) \]
\[ \quad = \tan (x \text{ radians}) = \tan (t \text{ degrees}) = \tan t^\circ \]
4) \[ y = \cot x = \frac{a}{b} \quad (b \neq 0) \]
\[ \quad = \cot (x \text{ radians}) = \cot (t \text{ degrees}) = \cot t^\circ \]
5) \( y = \sec x = \frac{1}{a} \quad (a \neq 0) \)
\[ = \sec (x \ \text{radians}) = \sec (t \ \text{degrees}) = \sec t^\circ \]

6) \( y = \csc x = \frac{1}{b} \quad (b \neq 0) \)
\[ = \csc (x \ \text{radians}) = \csc (t \ \text{degrees}) = \csc t^\circ \]

Remark 1: Definition 1 uses the standard function notation, \( y = f(x) \), with \( f \) replaced by the name of a particular trigonometric function. For example, \( y = \cos x \) actually means \( y = \cos (x) \) and \( \cos t^\circ \) actually means \( \cos (t^\circ) \).

Remark 2: Remember that \( t^\circ = t \times \frac{\pi}{180} \ \text{radians} \) and
\[ x \ \text{radians} = (x \times \frac{180}{\pi})^\circ \]

Theorem 1:
For any real number \( x \) we have the following trigonometric identities:

1) \( \csc x = \frac{1}{\sin x} \).
2) \( \sec x = \frac{1}{\cos x} \).
3) \( \cot x = \frac{1}{\tan x} \).
4) \( \tan x = \frac{\sin x}{\cos x} \).
5) \( \cot x = \frac{\cos x}{\sin x} \).
6) \( \sin (-x) = -\sin (x) \).
7) \( \cos (-x) = \cos (x) \).
8) \( \tan (-x) = -\tan (x) \).
9) \( \cot (-x) = -\cot (x) \).
10) \( \sin^2 x + \cos^2 x = 1 \).
11) \( \sec^2 x = \tan^2 x + 1 \).
12) \( \csc^2 x = \cot^2 x + 1 \).
S 2.5: Graphs of Sine and Cosine Functions

2.5.1: Table for values of \( \sin x, \cos x, \text{ and } \tan x \) for selected values of \( x \)

<table>
<thead>
<tr>
<th>Values of ( x )</th>
<th>Degrees</th>
<th>0</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radians</td>
<td></td>
<td>0</td>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{\pi}{3} )</td>
<td>( \frac{\pi}{2} )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td></td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>1</td>
</tr>
<tr>
<td>( \cos x )</td>
<td></td>
<td>1</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>( \tan x )</td>
<td></td>
<td>0</td>
<td>( \frac{1}{\sqrt{3}} )</td>
<td>1</td>
<td>( \sqrt{3} )</td>
<td>Undefined</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Values of ( x )</th>
<th>Degrees</th>
<th>120</th>
<th>135</th>
<th>150</th>
<th>180</th>
<th>270</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radians</td>
<td></td>
<td>( \frac{2\pi}{3} )</td>
<td>( \frac{3\pi}{4} )</td>
<td>( \frac{5\pi}{6} )</td>
<td>( \pi )</td>
<td>( \frac{3\pi}{2} )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td></td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>(-1 )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td></td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( -\frac{\sqrt{3}}{2} )</td>
<td>(-1 )</td>
<td>0</td>
</tr>
<tr>
<td>( \tan x )</td>
<td></td>
<td>( -\sqrt{3} )</td>
<td>(-1 )</td>
<td>( -\frac{1}{\sqrt{3}} )</td>
<td>0</td>
<td>Undefined</td>
</tr>
</tbody>
</table>

**Definition:** A function \( f \) is periodic if there exists a positive real number \( p \) such that \( f(x) = f(x + p) \) for all \( x \) in the domain of \( f \). The smallest such positive number \( p \) is the period of \( f \).

**Remarks:**

1) The functions \( \sin x, \cos x, \sec x, \text{ and } \csc x \) are periodic functions with period \( 2\pi \).

2) The functions \( \tan x \) and \( \cot x \) are periodic functions with period \( \pi \).
2.5.2: The Graph of $\sin x$

The graph of the function $y = \sin x$ is the line passing through all the points $(x, \sin x)$ on the $xy$-plane.

The graph of the function $y = \sin x$ for the interval $[0, 2\pi]$ is the line passing through the points $(0, 0), \left(\frac{\pi}{6}, \frac{1}{2}\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{6}, \frac{1}{2}\right), (\pi, 0), \left(\frac{7\pi}{6}, -\frac{1}{2}\right), \left(\frac{3\pi}{2}, -1\right), \left(\frac{11\pi}{6}, -\frac{1}{2}\right), \text{ and } (2\pi, 0)$ which is shown in the following figure.

![Graph of $y = \sin x$](image)

The graph of the function $y = \sin x$ is shown in the following figure.

![Graph of $y = \sin x$](image)

The period of the function $y = \sin x$ is $2\pi$. The domain of the function $y = \sin x$ is the set of all real numbers $\mathbb{R}$.

The range of the function $y = \sin x$ is the interval $[-1, 1]$. 
2.5.3: The Graph of $\cos x$

The graph of the function $y = \cos x$ is the line passing through all the points $(x, \cos x)$ on the $xy$-plane.

The graph of the function $y = \cos x$ for the interval $[0, 2\pi]$ is the line passing through the points $(0, 1), \left(\frac{\pi}{3}, \frac{1}{2}\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{2\pi}{3}, -\frac{1}{2}\right), (\pi, -1), \left(\frac{4\pi}{3}, -\frac{1}{2}\right), \left(\frac{3\pi}{2}, 0\right), \left(\frac{5\pi}{3}, \frac{1}{2}\right)$, and $(2\pi, 1)$ which is shown in the following figure.

![Graph of cos x](image)

The graph of the function $y = \cos x$ is shown in the following figure.

![Graph of cos x](image)

The period of the function $y = \cos x$ is $2\pi$.

The domain of the function $y = \cos x$ is the set of all real numbers $\mathbb{R}$.

The range of the function $y = \cos x$ is the interval $[-1, 1]$. 

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أضفت المادة: جان فاضل
2.5.4: The Graphs of $\tan x$ and $\sec x$

The graph of the function $y = \tan x$ is the line passing through all the points $(x, \tan x)$ on the $xy$-plane.

The graph of $y = \tan x$ is shown in the following figure:

![Graph of $\tan x$]

The graph of $y = \sec x$ is shown in the following figure:

![Graph of $\sec x$]
Exercise: Draw the graph of the following trigonometric functions:

1) $y = \csc(x)$

2) $y = \cot(x)$
CH3: Limits, Continuity and Differentiation

S3.1: Limits and Continuity

Remark 3.1.1: If the values of a function \( y = f(x) \) can be made as close as we like to a fixed number \( L \) by taking \( x \) close to \( x_0 \) (but not equal to \( x_0 \)), we say that \( L \) is the limit of \( f \) as \( x \) approaches \( x_0 \), and we write it as

\[
\lim_{x \to x_0} f(x) = L
\]

Also we can say that the limit of \( f \) as \( x \) approaches \( x_0 \) equals \( L \).

Definition 3.1.2:

Let \( f \) be a function defined on the set \((x_0 - p, x_0) \cup (x_0, x_0 + p)\), with \( p > 0 \). Then

\[
\lim_{x \to x_0} f(x) = L
\]

iff for each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

if \( 0 < |x - x_0| < \delta \) then \( |f(x) - L| < \varepsilon \).

Theorem 1:

1) \( \lim_{x \to x_0} x = x_0 \)

2) \( \lim_{x \to x_0} k = k \)

Theorem 2: If \( \lim_{x \to x_0} f(x) = L_1 \) and \( \lim_{x \to x_0} g(x) = L_2 \), then

1) \( \lim_{x \to x_0} [f(x) + g(x)] = L_1 + L_2 \)

2) \( \lim_{x \to x_0} [f(x) - g(x)] = L_1 - L_2 \)

3) \( \lim_{x \to x_0} [f(x) \cdot g(x)] = L_1 \cdot L_2 \)

4) \( \lim_{x \to x_0} [k \cdot f(x)] = k \cdot L_1 \)

5) \( \lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2} \) if \( L_2 \neq 0 \).
Example 3.1.3: Find each of the following:

1. \( \lim_{x \to 2} 7 \)
2. \( \lim_{x \to 1} x(3-x) \)
3. \( \lim_{x \to 3} (x^2 + 2x - 1) \)
4. \( \lim_{x \to 2} \frac{x-2}{x^2 - 5x + 6} \)
5. \( \lim_{x \to 0} \frac{x^2 - 5x}{x} \)

Solution:

1. \( \lim_{x \to 2} 7 = 7 \)
2. \( \lim_{x \to 1} x(3-x) = 1(3-1) = 2 \)
3. \( \lim_{x \to 3} (x^2 + 2x - 1) = (3)^2 + 2(3) - 1 = 9 + 6 - 1 = 14 \)
4. \( \lim_{x \to 2} \frac{x-2}{x^2 - 5x + 6} = \lim_{x \to 2} \frac{x-2}{(x-3)(x-2)} = \lim_{x \to 2} \frac{1}{x-3} = \frac{1}{2-3} = -1 \)
5. \( \lim_{x \to 0} \frac{x^2 - 5x}{x} = \lim_{x \to 0} \frac{x(x-5)}{x} = \lim_{x \to 0} (x-5) = 0 - 5 = -5 \)

Theorem 3:

1) \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)
2) \( \lim_{x \to 0} \frac{1 - \cos x}{x} = 0 \)

Example 3.1.4: Find each of the following:

1. \( \lim_{x \to 0} \frac{\sin 4x}{\sin 5x} \)
2. \( \lim_{x \to 0} \frac{3x}{\sin 2x} \)
3. \( \lim_{x \to 0} \frac{\tan x}{x} \)

**Solution:**

1. \( \lim_{x \to 0} \frac{\sin 4x}{\sin 5x} = \lim_{x \to 0} \frac{4x}{5x} \cdot \frac{\sin 4x}{\sin 5x} = \frac{4}{5} \)

2. \( \lim_{x \to 0} \frac{3x}{\sin 2x} = \lim_{x \to 0} \frac{3x}{2x} \cdot \frac{\sin 2x}{2x} = \frac{3}{2} \)

3. \( \lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{\cos x} = \lim_{x \to 0} \left( \frac{\sin x}{x} \cdot \frac{1}{\cos x} \right) = \lim_{x \to 0} \left( \frac{\sin x}{x} \cdot \frac{1}{\cos x} \right) = 1 \times 1 = 1 \)

**Exercise 3.1.5:** Find each of the following:

1. \( \lim_{x \to 0} \frac{1 - \cos x}{x + \sin x} \)

2. \( \lim_{x \to \infty} \left( 1 + \cos \frac{1}{x} \right) \)

3. \( \lim_{x \to 0} \frac{\sin 2x}{2x^2 + x} \)

4. \( \lim_{y \to 0} \frac{\tan 2y}{3y} \)

5. \( \lim_{y \to \infty} \frac{y^4}{y^4 - 7y^3 + 3y^2 + 9} \)

**Definition 3.1.6:** A function \( f(x) \) is said to be continuous at \( x_0 \) if

1) \( f \) is defined at \( x_0 \) (i.e. \( f(x_0) = L \) where \( L \in \mathbb{R} \)).

2) \( \lim_{x \to x_0} f(x) \) exists

3) \( \lim_{x \to x_0} f(x) = f(x_0) = L \)
Example 3.1.7: Let \( f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 3 - 2x & \text{if } x > 1 \end{cases} \)

Is \( f \) continuous at \( x = 1 \)?

Solution:
1) \( f(1) = 1^2 = 1 \)
2) \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1^2 = 1 \)
   \( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3 - 2x) = 3 - 2(1) = 1 \)
   since \( \lim_{x \to 1^-} f(x) = 1 = \lim_{x \to 1^+} f(x) \).
   Therefore \( \lim_{x \to 1^-} f(x) \) exists and \( \lim_{x \to 1^+} f(x) = 1 \)
3) \( \lim_{x \to 1^-} f(x) = 1 = f(1) \)
   Therefore \( f \) is continuous at \( x = 1 \).

Example 3.1.8: Let \( f(x) = \begin{cases} 2x + 1 & \text{if } x < -2 \\ x^2 - 2 & \text{if } x \geq -2 \end{cases} \)

Is \( f \) continuous at \( x = -2 \)?

Solution:
1) \( f(-2) = (-2)^2 - 2 = 4 - 2 = 2 \)
2) \( \lim_{x \to (-2)^-} f(x) = \lim_{x \to (-2)^-} (2x + 1) = 2(-2) + 1 = -4 + 1 = -3 \)
   \( \lim_{x \to (-2)^+} f(x) = \lim_{x \to (-2)^+} (x^2 - 2) = (-2)^2 - 2 = 4 - 2 = 2 \)
   since \( \lim_{x \to (-2)^-} f(x) \neq \lim_{x \to (-2)^+} f(x) \).
   Therefore \( \lim_{x \to (-2)} f(x) \) does not exists.
   Thus \( f \) is not continuous at \( x = -2 \).
Exercise 3.1.9:

Let \( f(x) = \begin{cases} 
\frac{x^2-2x-8}{x+2} & \text{if } x \neq -2 \\
-3 & \text{if } x = -2
\end{cases} \)

Is \( f \) continuous at \( x = -2 \)?

S3.2: Differentiation

Definition of Derivative, Rules of Differentiation

Definition 3.2.1:

Let \( y = f(x) \) be a function and let the variable \( x \) receive a certain increment \( \Delta x \). Then the function \( y \) will receive a certain increment \( \Delta y \). Thus for the value of \( x \) we have \( y = f(x) \) and for the value of \( x + \Delta x \), we have \( y + \Delta y = f(x + \Delta x) \).

Thus the increment \( \Delta y \) is given by:

\[
\Delta y = f(x + \Delta x) - f(x)
\]

Remark 3.2.2: \( \Delta \) is an abbreviation of difference (in \( x, y \)) and is not a factor.

Forming the ratio of the increment of the function \( y \) to the increment of the variable \( x \), we get

\[
\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

is called the average rate of change of the function \( y = f(x) \) with respect to the variable \( x \). \( \frac{\Delta y}{\Delta x} \) is also called the difference quotient of the function \( y = f(x) \). If the limit of this ratio as \( \Delta x \) approaches zero exists, that is

\[
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

exist, then the function is called differentiable and the limit \( \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \) is called the first derivative of the function \( y = f(x) \) with respect to
the variable \( x \), which is denoted by \( f'(x) \), \( y' \), \( \frac{dy}{dx} \), \( \frac{d}{dx} y \), \( \frac{d}{dx} f(x) \).

**Differentiation Rules:**

Let \( f(x) \) and \( g(x) \) be two differentiable functions (in the interval under consideration), then

---

**RULE 1  Constant Multiple Rule**

If \( f(x) \) is a differentiable function of \( x \), and \( c \) is a constant, then

\[
\frac{d}{dx} (cf(x)) = c \frac{d}{dx} f(x)
\]

---

**RULE 2  Derivative of the Sum**

If \( f(x) \) and \( g(x) \) are differentiable functions of \( x \), then their sum \( f(x) + g(x) \) is differentiable, and

\[
\frac{d}{dx} (f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)
\]

---

**RULE 3  Derivative of the Difference**

If \( f(x) \) and \( g(x) \) are differentiable functions of \( x \), then their difference \( f(x) - g(x) \) is differentiable, and

\[
\frac{d}{dx} (f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)
\]

---

**RULE 4  Derivative of the Product**

If \( f(x) \) and \( g(x) \) are differentiable functions of \( x \), then their product \( f(x) \cdot g(x) \) is differentiable, and

\[
\frac{d}{dx} (f(x) \cdot g(x)) = f(x) \cdot \frac{d}{dx} g(x) + g(x) \cdot \frac{d}{dx} f(x)
\]
Derivative of the Quotient

If \( f(x) \) and \( g(x) \) are differentiable functions of \( x \) and \( g(x) \neq 0 \), then the quotient \( \frac{f(x)}{g(x)} \) is differentiable, and

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{g(x) \cdot \frac{d}{dx} f(x) - f(x) \cdot \frac{d}{dx} g(x)}{g(x)^2}
\]

Derivatives of Some Special Functions and the Chain Rule:

1) Derivatives of Some Algebraic Functions:

1) Derivative of a Constant Function

If \( f(x) = c \), then

\[
\frac{d}{dx} f(x) = \frac{d}{dx} c = 0
\]

Example 3.2.3: If \( f(x) = 12 \), then

\[
\frac{d}{dx} f(x) = \frac{d}{dx} (12) = 0
\]

2) Derivatives of a Power Functions

\[
\frac{d}{dx} x^n = nx^{n-1}, \quad n \in \mathbb{Q}
\]

provided that \( x \neq 0 \) when \( n \) is negative.

Example 3.2.4: Find \( f' \) for each of the following functions:

(i) \( f(x) = x \),  (ii) \( f(x) = x^2 \),  (iii) \( f(x) = x^{-3} \),  (iv) \( f(x) = x^{0.3} \)

Solution:

(i) \( f'(x) = x^{1-1} = x^0 = 1 \)

(ii) \( f'(x) = 2x^{2-1} = 2x \)

(iii) \( f'(x) = -3x^{-2-1} = -3x^{-4} \)

(iv) \( f'(x) = 0.3x^{0.3-1} = 0.3x^{-0.7} \)
Example 3.2.5: Find $f'$ for each of the following functions:

(i) $f(x) = \frac{1}{2}x$, (ii) $f(x) = 9x^2$, (iii) $f(x) = 4x^{-3}$, (iv) $f(x) = x^{2.5}$.

Solution:

(i) $f'(x) = \frac{1}{2}$

(ii) $f'(x) = 9 \cdot 2x^{2-1} = 18x$

(iii) $f'(x) = 4 \cdot (-3)x^{-3-1} = -12x^{-4}$

(iv) $f'(x) = 2.5x^{2.5-1} = 2.5x^{1.5}$

Example 3.2.6: Find $f'$ for each of the following functions:

(i) $f(x) = x^3 + 5x^{-3}$, (ii) $f(x) = x^4 - \frac{3}{5}x^4 + 7x - 14$

Solution:

(i) $f'(x) = 3x^2 - 15x^{-4}$

(ii) $f'(x) = 4x^3 - \frac{3 \cdot 2}{5}x^0 + 7 - 0 = 4x^3 - \frac{6}{5}x^0 + 7$

Example 3.2.7: Find $f'$ for the function $f(x) = 2x \left( 3x^2 + \frac{3}{x} \right)$

Solution:

$$f'(x) = 2x \left( 15x^4 - \frac{3}{x^2} \right) + \left( 3x^2 + \frac{3}{x} \right) \cdot 2$$

$$= 30x^5 - \frac{6}{x} + 6x^4 + \frac{6}{x} = 36x^5$$

Example 3.2.8: Find $f'$ for the function $f(x) = \frac{2x-1}{3x+1}$

Solution:

$$f'(x) = \frac{(3x+1) \cdot 2 - (2x-1) \cdot 3}{(3x+1)^2}$$

$$= \frac{6x+2-6x+3}{(3x+1)^2} = \frac{5}{(3x+1)^2}$$
The derivative of the cosine function is the negative of the sine function:
\[
\frac{d}{dx}(\cos x) = -\sin x
\]

**Example 3.2.10**: Find \( f'(x) \) for the function \( f(x) = 3x^2 + 2\cos x \)

**Solution**: \( f'(x) = 6x - 2\sin x \)

**Example 3.2.11**: Find \( y' \) for each of the following functions:

(i) \( y = \sin x - \cos x \)  
(ii) \( y = 2\sin x \cos x \)  
(iii) \( y = \frac{3\sin x}{\cos x + 1} \)

**Solution**:

(i) \( y' = \cos x + \sin x \)

(ii) \( y' = 2\sin x \cdot (-\sin x) + \cos x \cdot (2\cos x) = -2\sin^2 x + 2\cos^2 x \)

(iii) \( y' = \frac{(\cos x + 1) \cdot (3\cos x) - (3\sin x) \cdot (-\sin x)}{(\cos x + 1)^2} \)

\[= \frac{3\cos^2 x + 3\cos x + 3\sin^2 x}{(\cos x + 1)^2} \]

The derivative of other trigonometric functions:

\[
\frac{d}{dx}(\tan x) = \sec^2 x
\]

\[
\frac{d}{dx}(\cot x) = -\csc^2 x
\]

\[
\frac{d}{dx}(\sec x) = \sec x \tan x
\]

\[
\frac{d}{dx}(\csc x) = -\csc x \cot x
\]

**Example 3.2.12**: Find \( y' \) for each of the following functions:

(i) \( y = \tan x + \sec x \)  
(ii) \( y = 5\cot x \csc x \)
Solution:

(i) \[ y' = \sec^2 x + \sec x \tan x \]

(ii) \[ y' = 5\cot x \cdot (-\csc x \cot x) + \csc x \cdot (-5 \csc^2 x) \]
\[ = -5 \csc x \cot^2 x - 5 \csc^3 x \]

Derivative of Logarithmic Function:

The derivative of the natural logarithmic function is:

\[ \frac{d}{dx} (\ln x) = \frac{1}{x} \]

Example 3.2.13: Find \( y' \) for each of the following functions:

(i) \[ y = 4x^3 \ln x \]
(ii) \[ y = \frac{2\ln x}{9x+1} \]

Solution:

(i) \[ y' = 4x^3 \left( \frac{1}{x} \right) + \ln x (12x^2) = 4x^2 + 12x^2 \ln x \]

(ii) \[ y' = \frac{\left(9x+1\right) \left(\frac{2}{x}\right)-(2\ln x)(9)}{(9x+1)^2} = \frac{18 + \frac{2}{x} - 18\ln x}{(9x+1)^2} \]

Derivative of Exponential Function:

The derivative of the exponential functions are:

\[ \frac{d}{dx} a^x = a^x \ln a \quad \text{and} \quad \frac{d}{dx} e^x = e^x \]

Example 3.2.14: Find \( f' \) for the function \( f(x) = 5x^7 e^x + 4 e^x \).

Solution: \[ f'(x) = 5x^7 e^x + e^x \left( 35x^6 \right) + 4e^x = 5x^7 e^x + 35x^6 e^x + 4e^x \]
Implicit Differentiation (Derivative of Composite Functions):

Chain Rule:

Let \( y = f(u) \), \( u = g(x) \) then \( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \)

Example 3.2.15: Let \( y = 6u^3 + 5u \), \( u = \ln x \), find \( \frac{dy}{dx} \).

Solution:

\[
\frac{dy}{du} = 18u^2 + 5, \quad \frac{du}{dx} = \frac{1}{x} \\
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (18u^2 + 5) \left( \frac{1}{x} \right) = \left( 18(\ln x)^3 + 5 \right) \left( \frac{1}{x} \right) \\
= \frac{18}{x}(\ln x)^3 + \frac{5}{x}
\]

Example 3.2.16: Find \( \frac{dy}{dx} \) for each of the following functions:

(i) \( y = (x + 4x^3)^6 \), (ii) \( y = \ln(x^2 + 3) \), (iii) \( y = \tan^3 x \).

Solution:

(i) let \( u = x + 4x^3 \), then \( y = u^6 \).

Thus \( \frac{dy}{du} = 6u^5 \) and \( \frac{du}{dx} = 1 + 12x^2 \)

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 6u^5(1 + 12x^2) = 6(x + 4x^3)^5(1 + 12x^2)
\]

(ii) let \( u = x^2 + 3 \), then \( y = \ln u \).

Thus \( \frac{dy}{du} = \frac{1}{u} \) and \( \frac{du}{dx} = 2x \)

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u}(2x) = \frac{2x}{x^2 + 3}
\]

(iii) let \( u = \tan x \), then \( y = u^3 \).

Thus \( \frac{dy}{du} = 3u^2 \) and \( \frac{du}{dx} = \sec^2 x \)

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2(\sec^2 x) = 3\tan^2 x \sec^2 x
\]
In examples (3.2.15 and 3.2.16) we use the Chain rule to get the derivative of a composite function using substitutions, but also we can get the same results directly without substitutions, considering the following rules:

\[
\frac{d}{dx} (f(x))^n = n (f(x))^{n-1} \cdot f'(x),
\]

\[
\frac{d}{dx} \ln f(x) = \frac{1}{f(x)} \cdot f'(x),
\]

\[
\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x),
\]

\[
\frac{d}{dx} \sin(f(x)) = \cos(f(x)) \cdot f'(x),
\]

\[
\frac{d}{dx} \cos(f(x)) = -\sin(f(x)) \cdot f'(x),
\]

\[
\frac{d}{dx} \tan(f(x)) = \sec^2(f(x)) \cdot f'(x),
\]

\[
\frac{d}{dx} \sec(f(x)) = \sec(f(x)) \cdot \tan(f(x)) \cdot f'(x),
\]

\[
\frac{d}{dx} \csc(f(x)) = -\csc(f(x)) \cdot \cot(f(x)) \cdot f'(x),
\]

\[
\frac{d}{dx} \cot(f(x)) = -\csc^2(f(x)) \cdot f'(x).
\]

**Example 3.2.17**: Find \( \frac{dy}{dx} \) for each of the following functions:

(i) \( y = \sqrt{x^5 + 4x} \)

(ii) \( y = \ln(x^2 + 3x) \)

(iii) \( y = e^{2x} \)

**Solution**:

(i) \( y = \sqrt{x^5 + 4x} = (x^5 + 4x)^{\frac{1}{2}} \)

\[
\frac{dy}{dx} = \frac{1}{2} (x^5 + 4x)^{-\frac{1}{2}} \cdot (5x^4 + 4) = \frac{5x^4 + 4}{2\sqrt{x^5 + 4x}}.
\]
Solution:

(i) \( y' = 20x^4 - 21x^3 + 3 \), \( y'' = 80x^3 - 42x \)

(ii) \( y' = x^3 (4e^{4x}) + e^{4x} (3x^2) = 4x^3e^{4x} + 3x^2e^{4x} \)
    \[ y'' = 4x^3 (4e^{4x}) + e^{4x} (12x^2) + 3x^2 (4e^{4x}) + e^{4x} (6x) \]
    \[ = 16x^3e^{4x} + 12x^2e^{4x} + 12xe^{4x} + 6xe^{4x} \]
    \[ = 16x^3e^{4x} + 24x^2e^{4x} + 6xe^{4x} \]

(iii) \( y' = 2\cos x - 9\sin x \)
    \[ y'' = -2\sin x - 9\cos x \]

Example 3.2.19: Find \( y' \), \( y'' \), \( y''' \) and \( y^{(iv)} \) for each of the following functions:

(i) \( y = x^6 + x^4 - 3x^3 \), (ii) \( y = e^{2x} \), (iii) \( y = \sin x \), (iv) \( y = \cos x \)

Solution:

(i) \( y' = 6x^5 + 4x^3 - 9x^2 \), \( y'' = 30x^4 + 12x^3 - 18x \)
    \[ y''' = 120x^3 + 24x - 18 \), \( y^{(iv)} = 360x^2 + 24 \).

(ii) \( y' = 2e^{2x} \), \( y'' = 4e^{2x} \), \( y''' = 8e^{2x} \), \( y^{(iv)} = 16e^{2x} \)

(iii) \( y' = \cos x \), \( y'' = -\sin x \), \( y''' = -\cos x \), \( y^{(iv)} = \sin x \)

(iv) \( y' = -\sin x \), \( y'' = -\cos x \), \( y''' = \sin x \), \( y^{(iv)} = \cos x \)

S3.3: L'Hopital Rule

Suppose that \( f(x_0) = g(x_0) = 0 \), and both \( f'(x_0) \) and \( g'(x_0) \) exist. Then

\[ \lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)} \quad \text{if} \quad g'(x_0) \neq 0. \]
(ii) $y = \ln(x^2 + 3x)$

\[
\frac{dy}{dx} = \frac{1}{x^2 + 3x} \cdot (2x + 3) = \frac{2x + 3}{x^2 + 3x}.
\]

(iii) $y = e^{3x}$

\[
\frac{dy}{dx} = e^{3x} \cdot 3 = 3e^{3x}.
\]

Second Order Derivative and Derivatives of Higher Order:

When we differentiate a function $y = f(x)$ we get a new function $y'$ (or $\frac{dy}{dx}$ or $f'(x)$ or $\frac{d}{dx}f$) which is the derivative of $y = f(x)$ (or the first derivative of $y = f(x)$). Now if this derivative $y' = f'(x)$ is also a differentiable function, we can define the second derivative of $y = f(x)$ (or the second order derivative of $y = f(x)$) by differentiating $y'$ (or $\frac{dy}{dx}$ or $f'(x)$ or $\frac{d}{dx}f$), which is denoted by $y''$ (or $\frac{d^2y}{dx^2}$ or $f''(x)$ or $\frac{d^2}{dx^2}f$).

Now if the second derivative $y'' = f''(x)$ is also a differentiable function, we can define the third derivative of $y = f(x)$ (or the third order derivative of $y = f(x)$) by differentiating $y''$ (or $\frac{d^2y}{dx^2}$ or $f''(x)$ or $\frac{d^2}{dx^2}f$), which is denoted by $y'''$ (or $\frac{d^3y}{dx^3}$ or $f'''(x)$ or $\frac{d^3}{dx^3}f$). So long as we have differentiability, we can continue in this manner forming the fourth derivative of $y = f(x)$, which is denoted by $y^{(4)}$ (or $\frac{d^4y}{dx^4}$ or $f^{(4)}(x)$ or $\frac{d^4}{dx^4}f$), and more generally the $n$th derivative of $y = f(x)$ is denoted by $y^{(n)}$ (or $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$ or $\frac{d^n}{dx^n}f$).

Example 3.2.18: Find $y''$ for each of the following functions:

(i) $y = 4x^5 - 7x^3 + 3x$,  
(ii) $y = x^3 e^{4x}$,  
(iii) $y = 2\sin x + 9\cos x$
(ii) \( y = \ln ( x^2 + 3x ) \)
\[
\begin{align*}
\frac{dy}{dx} &= \frac{1}{x^2 + 3x} \cdot (2x + 3) = \frac{2x + 3}{x^2 + 3x}.
\end{align*}
\]

(iii) \( y = e^{3x} \)
\[
\begin{align*}
\frac{dy}{dx} &= e^{3x} \cdot 3 = 3e^{3x}.
\end{align*}
\]

Second Order Derivative and Derivatives of Higher Order:

When we differentiate a function \( y = f(x) \) we get a new function \( y' \)
( or \( \frac{dy}{dx} \) or \( f'(x) \) or \( \frac{d}{dx} f \) ) which is the derivative of \( y = f(x) \) ( or the first derivative of \( y = f(x) \) ). Now if this derivative \( y' = f'(x) \) is also a differentiable function, we can define the second derivative of \( y = f(x) \) ( or the second order derivative of \( y = f(x) \) ) by differentiating \( y' \) ( or \( \frac{dy}{dx} \) or \( f'(x) \) or \( \frac{d}{dx} f \)), which is denoted by \( y'' \) ( or \( \frac{d^2y}{dx^2} \) or \( f''(x) \) or \( \frac{d^2}{dx^2} f \)).

Now if the second derivative \( y'' = f''(x) \) is also a differentiable function, we can define the third derivative of \( y = f(x) \) ( or the third order derivative of \( y = f(x) \) ) by differentiating \( y'' \) ( or \( \frac{d^2y}{dx^2} \) or \( f''(x) \) or \( \frac{d^2}{dx^2} f \)), which is denoted by \( y''' \) ( or \( \frac{d^3y}{dx^3} \) or \( f'''(x) \) or \( \frac{d^3}{dx^3} f \)). So long as we have differentiability, we can continue in this manner forming the fourth derivative of \( y = f(x) \), which is denoted by \( y^{(4)} \)
( or \( \frac{d^4y}{dx^4} \) or \( f^{(4)}(x) \) or \( \frac{d^4}{dx^4} f \)), and more generally the nth derivative of \( y = f(x) \) is denoted by \( y^{(n)} \) ( or \( \frac{d^ny}{dx^n} \) or \( f^{(n)}(x) \) or \( \frac{d^n}{dx^n} f \)).

Example 3.2.18: Find \( y'' \) for each of the following functions:

(i) \( y = 4x^5 - 7x^2 + 3x \),
(ii) \( y = x^4 e^{4x} \),
(iii) \( y = 2\sin x + 9\cos x \)
Solution:

(i) \( y' = 20x^4 - 21x^2 + 3 \), \( y'' = 80x^3 - 42x \)

(ii) \( y' = x^3 (4e^{2x}) + e^{2x} (3x^2) = 4x^3 e^{2x} + 3x^2 e^{2x} \)
\[ y'' = 4x^3 (4e^{2x}) + e^{2x} (12x^2) + 3x^2 (4e^{2x}) + e^{2x} (6x) \]
\[ = 16x^3 e^{2x} + 12x^2 e^{2x} + 12x^2 e^{2x} + 6x e^{2x} \]
\[ = 16x^3 e^{2x} + 24x^2 e^{2x} + 6x e^{2x} \]

(iii) \( y' = 2 \cos x - 9 \sin x \)
\( y'' = -2 \sin x - 9 \cos x \)

Example 3.2.19: Find \( y' \), \( y'' \), \( y''' \) and \( y^{(iv)} \) for each of the following functions:

(i) \( y = x^6 + x^4 - 3x^3 \), (ii) \( y = e^{2x} \), (iii) \( y = \sin x \), (iv) \( y = \cos x \)

Solution:

(i) \( y' = 6x^5 + 4x^3 - 9x^2 \), \( y'' = 30x^4 + 12x^2 - 18x \)
\( \), \( y''' = 120x^3 + 24x - 18 \), \( y^{(iv)} = 360x^2 + 24 \).

(ii) \( y' = 2e^{2x} \), \( y'' = 4e^{2x} \), \( y''' = 8e^{2x} \), \( y^{(iv)} = 16e^{2x} \)

(iii) \( y' = \cos x \), \( y'' = - \sin x \), \( y''' = - \cos x \), \( y^{(iv)} = \sin x \)

(iv) \( y' = - \sin x \), \( y'' = - \cos x \), \( y''' = \sin x \), \( y^{(iv)} = \cos x \)

S3.3: L'Hopital Rule

Suppose that \( f(x_0) = g(x_0) = 0 \), and both \( f'(x_0) \) and \( g'(x_0) \) exist. Then
\[ \lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)} \text{ if } g'(x_0) \neq 0. \]
Example 3.3.1: Find each of the following limits by using L'Hopital rule:

1. \[ \lim_{x \to 1} \frac{x^3 - 1}{4x^3 - x - 3} \]
2. \[ \lim_{x \to 0} \frac{1 - \cos x}{x + x^2} \]
3. \[ \lim_{x \to 0} \frac{3x - \sin x}{x} \]
4. \[ \lim_{x \to 0} \frac{\sqrt{4 + x} - 2}{x} \]

Solution:

1. \[ \lim_{x \to 1} \frac{x^3 - 1}{4x^3 - x - 3} = \lim_{x \to 1} \frac{3x^2}{12x^2 - 1} = \frac{3}{12 - 1} = \frac{3}{11} \]

2. \[ \lim_{x \to 0} \frac{1 - \cos x}{x + x^2} = \lim_{x \to 0} \frac{0 + \sin x}{1 + 2x} = \frac{\sin 0}{1 + 0} = \frac{0}{1} = 0 \]

3. \[ \lim_{x \to 0} \frac{3x - \sin x}{x} = \lim_{x \to 0} \frac{3 - \cos x}{1} = \frac{3 - \cos 0}{1} = \frac{3 - 1}{1} = 2 \]

4. \[ \lim_{x \to 0} \frac{\sqrt{4 + x} - 2}{x} \]
   \[ = \lim_{x \to 0} \frac{1}{2 \sqrt{4 + x} + x} = \frac{1}{4} = \frac{1}{4} \]

Example 3.3.2: Find each of the following limits by using L'Hopital rule:

1. \[ \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \]
2. \[ \lim_{x \to 0} \frac{x^4 - 5x^2}{x^2 + x - \sin x} \]
Solution:

1. \( \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \)

\[
= \lim_{x \to 0} \frac{1 - \cos x}{x \cos x + \sin x}
\]

\[
= \lim_{x \to 0} \frac{\sin x}{-x \sin x + \cos x + \cos x} = \frac{0}{2} = 0
\]

Exercises:

In exercises 1 - 6, find \( y' \) and \( y'' \) (the first and second derivatives with respect to \( x \)).

1. \( y = x^3 + 6x - 5 \)
2. \( y = 3x^4 - \frac{6}{x^2} \)
3. \( y = 7x^2 - 3\sin x \)
4. \( y = 5\sin x \cos x \)
5. \( y = 3\tan x + 4\sec x \)
6. \( y = 2\sin x - 5\cos x \)

In exercises 7 - 9, find the first and second derivatives of the given function with respect to the given variable.

7. \( w = 2u^4 - 3u + 1 \)
8. \( y = 6t^4 - \frac{4}{t} \)
9. \( v = t^2 - 8\sin t \)
In exercises 10 - 12, find $y'$ by applying the Product Rule

10) $y = (4 + x)(x^3 - 2)$
11) $y = (x + 2)(x^3 + x - 4)$
12) $y = (4 + x)(x^2 - \frac{3}{x})$

In exercises 13 - 17, find $y'$.

13) $y = \tan x - 3\sin x$
14) $y = 5\sin 3x^2 + \sqrt{x}$
15) $y = 3\sin x - e^x$
16) $y = \frac{2\sin x}{3x}$
17) $y = \frac{2\tan x - 3x}{3x + 4}$

In exercises 18 - 21, find $y'$, $y''$, $y'''$, and $y^{(4)}$.

18) $y = x^5 + 6x^4 - 25x$
19) $y = 3\sin x$
20) $y = \cos 2x$
21) $y = e^{3x} + \ln x$

In exercises 22 - 24, find the limit by using L'Hopital rule.

22) $\lim_{x \to 1} \frac{x - 1}{3x^3 - x^2 - 2}$
23) $\lim_{x \to 0} \frac{\sin 5x}{x}$
24) $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$
Applications of Derivatives:

Slope and Tangent Line and Normal Line:

The slope of the curve $y = f(x)$ at any point $P(x, y)$ is $y' = f'(x)$.

The tangent line to the curve $y = f(x)$ at any point $P_0(x_0, f(x_0))$ is the line whose equation

$$\frac{y - f(x_0)}{x - x_0} = f'(x_0)$$

which pass through the point $P_0$ on the curve $y = f(x)$.

The normal line to the curve $y = f(x)$ at any point $P_0(x_0, f(x_0))$ is the line whose equation

$$\frac{y - f(x_0)}{x - x_0} = -\frac{1}{f'(x_0)}$$

which pass through the point $P_0$ on the curve $y = f(x)$.

Example 3.4.1: Find the slope of the curve of the function $y = f(x) = x^3 - 2x^2 + 4$ at the point $(1, 3)$. Then find the equation of each of the tangent line and the normal line to the curve at the point $(1, 3)$.

Solution:

The slope at any point $x = f'(x) = 3x^2 - 4x$

\[ x = f'(1) = 3 - 4 = -1 \]

\[ \frac{y - f(1)}{x - 1} = f'(1) \Rightarrow \frac{y - 3}{x - 1} = -1 \]

\[ y - 3 = -x + 1 \Rightarrow y + x - 4 = 0 \]

Thus the equation of the tangent line at the point $(1, 3)$ is $y + x - 4 = 0$.

\[ \frac{y - f(1)}{x - 1} = -\frac{1}{f'(1)} \Rightarrow \frac{y - 3}{x - 1} = 1 \]

\[ y - 3 = x - 1 \Rightarrow y - x - 2 = 0 \]

Thus the equation of the normal line at the point $(1, 3)$ is $y - x - 2 = 0$.

Example 3.4.2: Find the slope of the curve of the function $y = g(x) = x^2$ at the point $(3, 9)$. Then find the equation of each of the tangent line and the normal line to the curve at the point $(3, 9)$.
Solution: \( g'(x) = 2x \)

The slope of the curve at the point \((3, 9)\) is \( g'(3) = 2(3) = 6 \).

\[
\frac{y - g(3)}{x - 3} = g'(3) \quad \Rightarrow \quad \frac{y - 9}{x - 3} = 6 \quad \Rightarrow
\]

\[ y - 9 = 6x - 18 \quad \Rightarrow \quad y - 6x + 9 = 0 \]

Thus the equation of the tangent line at the point \((3, 9)\) is \( y - 6x + 9 = 0 \).

\[
\frac{y - g(3)}{x - 3} = \frac{1}{g'(3)} \quad \Rightarrow \quad \frac{y - 9}{x - 3} = \frac{1}{6} \]
\[
\Rightarrow \quad 6y - 54 = -x + 3 \quad \Rightarrow \quad 6y + x - 57 = 0
\]

Thus the equation of the normal line at the point \((3, 9)\) is \( 6y + x - 57 = 0 \).

Exercise 3.4.3: Find the slope of the curve of the function \( y = h(x) = 3x^2 - 1 \) at the point \((-1, 2)\). Then find the equation of each of the tangent line and the normal line to the curve at the point \((-1, 2)\).

Exercise 3.4.4: Find the slope of the curve of the function \( y = f(x) = x^3 - 4 \) at the point \((2, 4)\). Then find the equation of each of the tangent line and the normal line to the curve at the point \((2, 4)\).
S4.1 : The Indefinite Integral

Definition: A function \( F(x) \) is anti-derivative of a function \( f(x) \) with respect to \( x \) if \( \frac{d}{dx} F(x) = f(x) \) for all \( x \) in the domain of \( f \). The set of all anti derivatives of \( f \) is the indefinite integral of \( f \) with respect to \( x \), denoted by \( \int f(x) \, dx \) i.e. \( \int f(x) \, dx = F(x) + c \).

The symbol \( \int \) is an integral sign.

The function \( f \) is the integrand of the integral and \( x \) is the variable of the integration.

Example 4.1.1: \( \int 3x^2 \, dx = x^3 + c \).

Integral Formulas:

1) \( \int u^n \, du = \frac{u^{n+1}}{n+1} + c, \; n \neq -1, \; n \text{ rational} \)

\( \int du = \int 1 \, du = u + c \) (special case)

2) \( \int \sin u \, du = -\cos u + c \)

3) \( \int \cos u \, du = \sin u + c \)

4) \( \int \sec^2 u \, du = \tan u + c \)

5) \( \int \csc^2 u \, du = -\cot u + c \)

6) \( \int \sec u \tan u \, du = \sec u + c \)

7) \( \int \csc u \cot u \, du = -\csc u + c \)

8) \( \int \frac{1}{u} \, du = \ln |u| + c \)

9) \( \int e^u \, du = e^u + c \)

10) \( \int a^u \, du = \frac{a^u}{\ln a} + c \), \( a > 0 \)
Rules of Indefinite Integration:
1) \[ \int k f(x) \, dx = k \int f(x) \, dx \]
2) \[ \int - f(x) \, dx = - \int f(x) \, dx \]
3) \[ \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx \]

Example 4.1.2:
1) \[ \int dx = x + c \]
2) \[ \int x^5 \, dx = \frac{x^6}{6} + c \]
3) \[ \int \sin x \, dx = -\cos x + c \]
4) \[ \int \cos x \, dx = \sin x + c \]
5) \[ \int \sec^2 x \, dx = \tan x + c \]
6) \[ \int \csc^2 x \, dx = -\cot x + c \]
7) \[ \int \sec x \tan x \, dx = \sec x + c \]
8) \[ \int \csc x \cot x \, dx = -\csc x + c \]

Example 4.1.3: Find each of the following:
1) \[ \int (x^3 + 7)^4 \cdot 3x^2 \, dx \]
2) \[ \int (x^2 + 4x + 5)^{10} (x + 2) \, dx \]
3) \[ \int \sin(3x) \, dx \]
4) \[ \int 2x \sin(x^2) \, dx \]
5) \[ \int \sin^3 x \cos x \, dx \]
6) \[ \int 2 \cos 2x \, dx \]
7) \[ \int x^2 \cos(x^3) \, dx \]
8) \[ \int \sec^2(7x) \, dx \]
9) \[ \int \csc^2(6x) \, dx \]
10) \[ \int \csc(5x) \cot(5x) \, dx \]
Example 4.1.4: Find each of the following:

1) \[ \int \frac{1}{x} \, dx \]

2) \[ \int \tan x \, dx \]

3) \[ \int \cot x \, dx \]

4) \[ \int \frac{x+1}{x^2+3x+2} \, dx \]

5) \[ \int e^x \, dx \]

6) \[ \int 2x e^{x^2} \, dx \]

7) \[ \int (7x^2 - 5e^{7x}) \, dx \]

Solution:

1) \[ \int \frac{1}{x} \, dx = \ln|x| + c \]

2) \[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\ln|\cos x| + c \]

3) \[ \int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \ln|\sin x| + c \]

4) \[ \int \frac{x+1}{x^2+3x+2} \, dx = \int \frac{x+1}{(x+2)(x+1)} \, dx = \int \frac{1}{x+2} \, dx = \ln|x+2| + c \]

5) \[ \int e^x \, dx = e^x + c \]

6) \[ \int 2x e^{x^2} \, dx = e^{x^2} + c \]

7) \[ \int (7x^2 - 5e^{7x}) \, dx = \int 7x^2 \, dx - \int 5e^{7x} \, dx = \frac{7x^3}{3} - \frac{5e^{7x}}{7} + c \]

Exercise 4.1.5: Find each of the following:

1) \[ \int \cos^4 x \sin x \, dx \]

2) \[ \int \sec^2(3x) \, dx \]

3) \[ \int x^4 \sec^2(x^5) \, dx \]

4) \[ \int \sec^2 x \tan x \, dx \]

5) \[ \int \sec^2 x \tan^2 x \, dx \]

6) \[ \int \sec^4 x \tan x \, dx \]

7) \[ \int x^9 \csc^2(x^{10}) \, dx \]
S4.2 : The Definite Integral

Definition: If \( f \) is a continuous at every point of \([a, b]\) and if \( F \) is any anti-derivative of \( f \) on \([a, b]\), then

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

is called the definite integral.

Example 4.2.1: Evaluate the integral \( \int_1^4 (x^3 + 2x + 9) \, dx \)

Solution:

\[
\int_1^4 (x^3 + 2x + 9) \, dx = \left[ \frac{x^4}{4} + x^2 + 9x \right]^4_1
\]

\[
= \left( \frac{256}{4} + 16 + 36 \right) - \left( \frac{1}{4} + 1 + 9 \right)
\]

\[
= 116 - 10.25 = 105.75
\]

Example 4.2.2: Evaluate \( \int_0^\pi \sin x \, dx \)

Solution:

\[
\int_0^\pi \sin x \, dx = [-\cos x]^\pi_0 = 0 - (-1) = 1
\]

How to Find the Area:

To find the area between the graph of \( y = f(x) \) and the \( x \)-axis over the interval \([a, b]\), we should follow the following steps:

Step 1: Partition \([a, b]\) with the zeros of \( f \).

Step 2: Integrate \( f \) over each subinterval.

Step 3: Add the absolute values of the Integrals.

Example 4.2.3: Find the total area of the region between the curve \( y = x^2 + 2x \) and the \( x \)-axis over the interval \([-3, 4]\).

Solution:

\( x^2 + 2x = 0 \implies x(x + 2) = 0 \)

\( \Rightarrow x = 0 \) or \( x = -2 \)

\( \therefore \) the area

\[
= \left| \int_{-3}^{-2} (x^2 + 2x) \, dx \right| + \left| \int_{-2}^{0} (x^2 + 2x) \, dx \right| + \left| \int_{0}^{4} (x^2 + 2x) \, dx \right|
\]

\[
= \left| \left[ \frac{x^3}{3} + x^2 \right]_{-3}^{-2} \right| + \left| \left[ \frac{x^3}{3} + x^2 \right]_{-2}^{0} \right| + \left| \left[ \frac{x^3}{3} + x^2 \right]_{0}^{4} \right|
\]
\[ \begin{align*}
&= \left| \left( \frac{-8}{3} + 4 \right) - \left( \frac{-27}{3} + 9 \right) \right| + \left| (0 + 0) - \left( \frac{-8}{3} + 4 \right) \right| \\
&\quad + \left| \left( \frac{64}{3} + 16 \right) - (0 + 0) \right| \\
&= \frac{4}{3} + \frac{4}{3} + \frac{112}{3} = \frac{120}{3} = 40.
\end{align*} \]

**Example 4.2.4:** Find the total area of the region between the curve \( y = x^3 - 4x^2 + 3x \) and the \( x \)-axis over the interval \([0, 2]\).

**Solution:**
\[ x^3 - 4x^2 + 3x = 0 \implies x(x^2 - 4x + 3) = 0 \]
\[ \implies x(x - 1)(x - 3) = 0 \implies x = 0 \text{ (neglected)} \text{ or } x = 1 \]
\[ \text{or } x = 3 \text{ (neglected)} \]
\[ \text{the area} = \left| \int_0^1 (x^3 - 4x^2 + 3x) \, dx \right| + \left| \int_1^2 (x^3 - 4x^2 + 3x) \, dx \right| \]
\[ = \left| \left[ \frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2} \right]_0^1 \right| + \left| \left[ \frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2} \right]_1^2 \right| \]
\[ = \left| \left( \frac{1}{4} - \frac{4}{3} + \frac{3}{2} \right) - 0 \right| + \left| \left( \frac{16}{4} - \frac{32}{3} + \frac{12}{2} \right) - \left( \frac{1}{4} - \frac{4}{3} + \frac{3}{2} \right) \right| \]
\[ = \left| \frac{3-16+18}{12} \right| + \left| \frac{48-128+72}{12} - \frac{3-16+18}{12} \right| \]
\[ = \left| \frac{5}{12} \right| + \left| \frac{-8}{12} - \frac{5}{12} \right| = \frac{5}{12} + \left| -\frac{13}{12} \right| = \frac{5}{12} + \frac{13}{12} = \frac{18}{12} = 1.5 \]

**How to Find the Area Between Two Curves over an Interval \([a, b]\):**

To find the area between the two curves \( f(x) \) and \( g(x) \) over the interval \([a, b]\) we should follow the following steps:

**Step 1:** Partition \([a, b]\) with the zeros of \( f - g \).

**Step 2:** Integrate \( f - g \) over each subinterval.

**Step 3:** Add the absolute values of the Integrals.

**Example 4.2.5:** Find the total area of the region between the two curves \( f(x) = x^2 \) and \( g(x) = 2x \) over the interval \([-1, 2]\).

**Solution:**
\[ f(x) - g(x) = x^2 - 2x = 0 \implies x(x - 2) = 0 \]
\[ x = 0 \text{ or } x = 2 \text{ (neglected)} \]
the area $= \left| \int_{-1}^{0} (x^2 - 2x) \, dx \right| + \left| \int_{0}^{2} (x^2 - 2x) \, dx \right|$

$= \left| \left[ \frac{x^3}{3} - x^2 \right]_{-1}^{0} \right| + \left| \left[ \frac{x^3}{3} - x^2 \right]_{0}^{2} \right|$

$= \left| (0 - 0) - \left( \frac{1}{3} + 1 \right) \right| + \left| \left( \frac{8}{3} - 4 \right) - (0 - 0) \right|$

$= \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

**How to Find the Area Between Two Curves:**

To find the area between the two curves $f(x)$ and $g(x)$ we should follow the following steps:

**Step 1:** Find the zeros of $f - g$, and let them be $a$ and $b$.

**Step 2:** Integrate $f - g$ over the interval $[a, b]$.

**Step 3:** Find the absolute value of the integration found in step 2.

**Example 4.2.6:** Find the area of the region enclosed by the parabola $y = x^2 - 2$ and the line $y = x$.

**Solution:**

$f(x) - g(x) = (x^2 - 2) - x = 0 \Rightarrow x^2 - 2 - x = 0$

$\Rightarrow (x - 2)(x + 1) = 0 \Rightarrow x = 2 \text{ or } x = -1$.

$\therefore \text{the area} = \left| \int_{-1}^{2} (x^2 - 2 - x) \, dx \right|$

$= \left| \left[ \frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{-1}^{2} \right|$

$= \left| \left( \frac{8}{3} - \frac{4}{2} - 4 \right) - \left( \frac{1}{3} - \frac{1}{2} + 2 \right) \right|$

$= \left| \frac{8 - 6 - 12}{3} - \frac{2 - 3 + 12}{6} \right|$

$= \left| \frac{-10 - 7}{6} \right| = \left| \frac{-20 - 7}{6} \right| = \frac{27}{6} = 4 \frac{1}{2}$.

**Example 4.2.7:** Find the total area of the region enclosed by the parabola $f(x) = x^2$ and the line $g(x) = 2x$.

**Solution:**

$f(x) - g(x) = x^2 - 2x = 0 \Rightarrow x(x - 2) = 0$

$\Rightarrow x = 0 \text{ or } x = 2$
\[ \text{the area} = \left| \int_{0}^{2} (x^2 - 2x) \, dx \right| \]

\[ = \left| \left[ \frac{x^3}{3} - x^2 \right]_{0}^{2} \right| \]

\[ = \left| \left( \frac{8}{3} - 4 \right) - (0 - 0) \right| \]

\[ = \left| \frac{8 - 12}{3} \right| = \left| -\frac{4}{3} \right| = \frac{4}{3} \]

**Rules for definite integrals**

1) Order of integration:
\[ \int_{b}^{a} f(x) \, dx = -\int_{a}^{b} f(x) \, dx \]

2) Zero integration:
\[ \int_{a}^{a} f(x) \, dx = 0 \]

3) Constant multiple:
\[ \int_{a}^{b} k f(x) \, dx = k \int_{a}^{b} f(x) \, dx \quad \forall \, k \in R, \text{ and thus} \]
\[ \int_{a}^{b} f(x) \, dx = -\int_{a}^{b} f(x) \, dx \quad \text{for} \quad k = -1 . \]

4) Sum and difference:
\[ \int_{a}^{b} (f(x) \pm g(x)) \, dx = \int_{a}^{b} f(x) \, dx \pm \int_{a}^{b} g(x) \, dx \]

5) Additively:
\[ \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx = \int_{a}^{c} f(x) \, dx \]

**Example 4.2.8:** Suppose that
\[ \int_{-2}^{1} f(x) \, dx = 4 \quad , \quad \int_{1}^{3} f(x) \, dx = -3 \quad \text{and} \quad \int_{-2}^{1} h(x) \, dx = 6 . \text{ Find} \]

1) \[ \int_{3}^{1} f(x) \, dx \]

2) \[ \int_{-2}^{1} (2 f(x) + 5 h(x)) \, dx \]
3) \[ \int_{-2}^{3} f(x) \, dx \]

4) \[ \int_{-2}^{1} (3f(x) - 2h(x)) \, dx \]

**Solution:**

1) \[ \int_{3}^{1} f(x) \, dx = -\int_{1}^{3} f(x) \, dx = -(-3) = 3 \]

2) \[ \int_{-2}^{1} (2f(x) + 5h(x)) \, dx = \int_{-2}^{1} 2f(x) \, dx + \int_{-2}^{1} 5h(x) \, dx \]
   \[ = 2\int_{-2}^{1} f(x) \, dx + 5\int_{-2}^{1} h(x) \, dx \]
   \[ = 2(4) + 5(6) = 8 + 30 = 38 \]

3) \[ \int_{-2}^{3} f(x) \, dx = \int_{-2}^{1} f(x) \, dx + \int_{1}^{3} f(x) \, dx \]
   \[ = 4 + (-3) = 1 \]

4) \[ \int_{-2}^{1} (3f(x) - 2h(x)) \, dx = \int_{-2}^{1} 3f(x) \, dx - \int_{-2}^{1} 2h(x) \, dx \]
   \[ = 3\int_{-2}^{1} f(x) \, dx - 2\int_{-2}^{1} h(x) \, dx \]
   \[ = 3(4) - 2(6) = 12 - 12 = 0 \]

**Exercise 4.2.9:** Evaluate the following integrals:

1) \[ \int_{-2}^{3} (2x + 5) \, dx \]

2) \[ \int_{0}^{1} (x^2 + \sqrt{x}) \, dx \]

3) \[ \int_{0}^{\pi} (1 + \cos x) \, dx \]

4) \[ \int_{\frac{\pi}{2}}^{\pi} (8y^2 + \sin y) \, dy \]

5) \[ \int_{2}^{\frac{2}{x^2}} \, dx \]
CH 5: Sequences and Series

5.1.1: Arithmetic Sequence and Geometric Sequence

Definition 5.1.1: A sequence of numbers is a set of numbers arranged in a specific order.

Each number is called a term in the sequence. The first number is called the first term and will be denoted by \(a_1\), the second number is called the second term and will be denoted by \(a_2\), ..., the \(n\)th number is called the \(n\)th term and will be denoted by \(a_n\).

The sequence will be written as \(a_1, a_2, ..., a_n, ...\) and will be denoted by \(\{a_n\}\).

Definition 5.1.2: If \(\{a_n\}\) is a given sequence and \(S_n\) is defined by

\[S_1 = a_1,\]
\[S_2 = a_1 + a_2,\]
\[S_3 = a_1 + a_2 + a_3,\]
\[\vdots\]
\[S_n = a_1 + a_2 + ... + a_n = \sum_{i=1}^{n} a_i\]

Then the sequence \(\{S_n\}\) will be called an infinite series and will be written as \(\sum_{i=1}^{\infty} a_i\), and the terms \(S_n\) will be called the partial sum of the series.

Definition 5.1.3: A sequence of numbers in which each
term after the first term is obtained by adding a fixed number that is added is called the common difference and will be denoted by \( d \).

Example 5.1.4: \( 7, 10, 13, 16, 19, \ldots \) is an arithmetic sequence, since each term after the first term is obtained by adding 3 to the previous term. In this example we have \( d = 3 \) and \( a_1 = 7 \):

\[
\begin{align*}
    a_2 &= a_1 + d = 7 + 3 = 10 \\
    a_3 &= a_2 + d = 10 + 3 = 13 \\
    a_4 &= a_3 + d = 13 + 3 = 16 \\
    &
\end{align*}
\]

Example 5.1.5: Find the common difference \( d \) for the following arithmetic sequence \( 3, 9, 15, 21, 27, \ldots \)

Solution:

\[
\begin{align*}
    d &= 9 - 3 = 6 \\
    \text{or } d &= 15 - 9 = 6 \\
    \text{or } d &= 21 - 15 = 6 \\
    \text{or } d &= 27 - 21 = 6
\end{align*}
\]

Example 5.1.6: Write the first seven terms of the arithmetic sequence whose first term \( a_1 = 4 \) and common difference \( d = 9 \).

Solution: The first term \( a_1 = 4 \).

The second term \( a_2 = a_1 + d = 4 + 9 = 13 \).
The third term \( a_3 = a_2 + d = 13 + 9 = 22 \)
The fourth term \( a_4 = a_3 + d = 22 + 9 = 31 \)
The fifth term \( a_5 = a_4 + d = 31 + 9 = 40 \)
The sixth term \( a_6 = a_5 + d = 40 + 9 = 49 \)
The seventh term \( a_7 = a_6 + d = 49 + 9 = 58 \).

Exercise 5.1.7: Write the first nine terms of each of the following arithmetic sequences if you know that

1. \( a_1 = 3 \) and \( d = -2 \)
2. \( a_1 = 7 \) and \( d = 5 \)
3. \( a_2 = 10 \) and \( d = 4 \)
4. \( a_4 = 12 \) and \( a_5 = 17 \).

Remark 5.1.8: In the arithmetic sequence with the first term \( a_1 \) and common difference \( d \), the \( n \)th term \( a_n \) is given by \( a_n = a_1 + (n-1)d \).

Example 5.1.9: Find \( a_{10} \), \( a_{12} \) and \( a_n \) for the arithmetic sequence \(-2, 5, 12, \ldots\)

Solution:
\( a_1 = -2 \) and \( d = 5 - (-2) = 5 + 2 = 7 \).

\( a_{10} = -2 + (10-1) 	imes 7 = -2 + 9 	imes 7 = -2 + 63 = 61 \).

\( a_{12} = -2 + (12-1) 	imes 7 = -2 + 11 	imes 7 = -2 + 77 = 75 \).

\( a_n = -2 + (n-1) 	imes 7 = -2 + 7n - 7 = 7n - 9 \).

Exercise 5.1.10: For each of the following arithmetic sequences, find \( d \), \( a_{15} \), \( a_n \).
(1) 2, 5, 8, 11, ...
(2) 5, 9, 13, 17, ...
(3) -5, 6, 17, 28, ...

5.2: Arithmetic Series and Geometric Series

Definition 5.2.1: If \( \{a_n\} \) is an arithmetic sequence, then the corresponding series \( \sum_{i=1}^{\infty} a_i \) is called an arithmetic series and the terms \( S_n = \sum_{i=1}^{n} a_i \) is called the nth partial sum of the arithmetic series.

Theorem (1):

(i) The nth partial sum of an arithmetic series is \( S_n = \frac{n}{2} (a_1 + a_n) \).

(ii) The nth partial sum of an arithmetic series is \( S_n = na_1 + \frac{n(n-1)}{2}d \).

Example 5.2.2: Find the sum of the first 100 positive integers.

Solution: \( a_1 = 1 \) and \( a_{100} = 100 \)

\[ S_{100} = \frac{100}{2} (1 + 100) = 50 (101) = 5050 \]

Example 5.2.3: Find the sum of the first 20 terms of the arithmetic sequence \( \{10, 16, 22, \ldots \} \).
Solution:
\[ a_1 = 10 \text{ and } d = 16 - 10 = 6 \]
\[ S_{20} = 20 \times 10 + \frac{20(20-1)}{2} \times 6 = 200 + \frac{20(19)}{2} \times 6 \]
\[ = 200 + 190 \times 6 \]
\[ = 200 + 1140 = 1340. \]

Example 5.2.4: The sum of the first 16 terms of an arithmetic sequence is 80. If \( a_{16} = 20 \), find \( a_1 \) and \( d \).

Solution:
\[ S_{16} = \frac{16}{2} (a_1 + a_{16}) = 8 (a_1 + 20) = 8a_1 + 160 \]
\[ \Rightarrow 80 = 8a_1 + 160 \Rightarrow 8a_1 = 80 - 160 \]
\[ \Rightarrow 8a_1 = -80 \]
\[ \Rightarrow a_1 = \frac{-80}{8} = -10 \]
Since \( a_{16} = a_1 + (16-1)d \) then \( 20 = -10 + 15d \)
\[ \Rightarrow 15d = 30 \Rightarrow d = \frac{30}{15} = 2. \]

Example 5.2.5: Find the sum of the first 12 terms of the sequence 11, 18, 25, ...

Solution: \( a_1 = 11 \text{ and } d = 18 - 11 = 7 \).
The \( i \)th term \( a_i = a_1 + (i-1)d = 11 + (i-1) \times 7 \)
\[ = 11 + 7i - 7 = 7i + 4 \]
\[ \Rightarrow S_n = \sum_{i=1}^{n} (7i + 4) \]
\[ \Rightarrow S_{12} = \sum_{i=1}^{12} (7i + 4) = \frac{12}{2} (a_1 + a_{12}) = 6 (11 + 88) \]
\[ = 6 \times 99 = 594. \]

Exercise 5.2.6: Find each of the following sums:
Definition 5.2.7: A sequence of numbers in which each term after the first term is obtained by multiplying the previous term by a fixed nonzero real number is called a geometric sequence. The fixed nonzero real number that is multiplied is called the common ratio and will be denoted by $r$.

Example 5.2.8: $3, 6, 12, 24, 48, ...$ is a geometric sequence, since each term after the first term is obtained by multiplying the previous term by $2$. In this example we have $r = 2$ and

- $a_1 = 3$
- $a_2 = r \cdot a_1 = 2(3) = 6$
- $a_3 = r \cdot a_2 = 2(6) = 12$
- $a_4 = r \cdot a_3 = 2(12) = 24$

Example 5.2.9: Find the common ratio $r$ for the following geometric sequence

$7, 21, 63, 189, ...$

Solution: $r = \frac{21}{7} = 3$

or $r = \frac{63}{21} = 3$

or $r = \frac{189}{63} = 3$
Example 5.2.10: Write the first five terms of the geometric sequence whose first term $a_1 = 3$ and common ratio $r = -2$.

Solution: The first term $a_1 = 3$

The second term $a_2 = r \cdot a_1 = (\text{-}2) \times 3 = \text{-}6$

The third term $a_3 = r \cdot a_2 = (\text{-}2) \times (\text{-}6) = 12$

The fourth term $a_4 = r \cdot a_3 = (\text{-}2) \times 12 = \text{-}24$

The fifth term $a_5 = r \cdot a_4 = (\text{-}2) \times (\text{-}24) = 48$.

Exercise 5.2.11: Write the first seven terms of each of the following geometric sequence if you know that

1. $a_1 = \text{-}6$ and $r = 2$
2. $a_1 = 7$ and $r = \text{-}1$
3. $a_1 = 4$ and $r = 5$

Remark 5.2.12: In the geometric sequence with the first term $a_1$ and the common ratio $r$, the $n$th term $a_n$ is given by $a_n = r^{n-1} \times a_1$.

Example 5.2.13: Find $a_5$, $a_7$ and $a_n$ for the geometric sequence 5, 10, 20, ...

Solution:

$a_1 = 5$ and $r = \frac{10}{5} = 2$

$a_5 = 2^{5-1} \times 5 = 2^4 \times 5 = 16 \times 5 = 80$

$a_7 = 2^{7-1} \times 5 = 2^6 \times 5 = 64 \times 5 = 320$

$a_n = 2^{n-1} \times 5$.
Exercise 5.2.14: For each of the following geometric sequences, find $r$, $a_6$ and $a_n$.

1. $6, 18, 54, ...$
2. $7, -7, 7, ...$
3. $-2, 4, -8, ...$

Definition 5.2.15: If $\{a_n\}$ is a geometric sequence, then the corresponding series $\sum_{i=1}^{\infty} a_i$ is called a geometric series and the terms

$$S_n = \sum_{i=1}^{n} a_i = a_1 + r a_1 + r^2 a_1 + \ldots + r^{n-1} a_1$$

is called the $n$th partial sum of the geometric series.

Theorem (2):

(i) The $n$th partial sum of a geometric series is

$$S_n = \begin{cases} 
\frac{a_1 (1 - r^n)}{1 - r} & \text{if } r \neq 1 \\
na_1 & \text{if } r = 1 
\end{cases}$$

(ii) The $n$th partial sum of a geometric series is

$$S_n = \begin{cases} 
\frac{a_1 - r a_n}{1 - r} & \text{if } r \neq 1 \\
n a_1 & \text{if } r = 1 
\end{cases}$$

Example 5.2.16: Find the sum $S_7$ of the first seven terms of the geometric sequence (the 7th partial sum of
the geometric series) 4, 8, 18, ...

Solution:
\[ r = \frac{8}{4} = 2 \], \( a_1 = 4 \)

The sum of the first seven terms is
\[ S_7 = \frac{4(1-2^7)}{1-2} = \frac{4(1-128)}{-1} = \frac{4(-127)}{-1} \]
\[ = -508 \]
\[ = 508 \]

Example 5.2.17: Find \( \sum_{i=1}^{5} 7(4)^i \).

Solution: \( a_1 = 7(4)^1 = 28 \)
\[ a_2 = 7(4)^2 = 7(16) = 112 \]
\[ \therefore r = \frac{a_2}{a_1} = \frac{112}{28} = 4 \]
\[ \therefore \sum_{i=1}^{5} 7(4)^i = S_5 = \frac{28(1-4^5)}{1-4} = \frac{28(1-1024)}{-3} \]
\[ = \frac{28(-1023)}{-3} = \frac{-28644}{-3} = 9548 \]

Exercise 5.2.18: Find each of the following:

(1) \( \sum_{i=1}^{7} 2(3)^i \)
(2) \( \sum_{i=1}^{6} 6(2)^i \)
(3) \( \sum_{i=1}^{5} 9(-2)^i \)
(4) \( \sum_{i=1}^{10} 2^i \)
Definition 5.2.19: Let \( \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \ldots + a_n + \ldots \) be an infinite series and let \( \{S_n\} \), where \( S_n = a_1 + a_2 + \ldots + a_n \) for \( n = 1, 2, 3, \ldots \) be the sequence of partial sums of the infinite series.

If \( \lim_{n \to \infty} S_n \) exists and equals a number \( S \), the series is said to be convergent (and to converge to the value \( S \)) and \( S \) is called the sum of the infinite series \( \sum_{i=1}^{\infty} a_i \).

If \( \lim_{n \to \infty} S_n \) fails to exist or not a finite number, the series is divergent and has no sum.

Example 5.2.20: Find the sum of the infinite series \( \sum_{m=1}^{\infty} \frac{1}{2^m} \).

Solution: \( S_1 = \frac{1}{2} \),
\[ S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \],
\[ S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \],
\[ \vdots \]
\[ S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = \frac{2^n - 1}{2^n} \]
\[ \vdots \]
Then \( \sum_{m=1}^{\infty} \frac{1}{2^m} = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{2^n - 1}{2^n} = \lim_{n \to \infty} (1 - \frac{1}{2^n}) \)
\[ = 1 - 0 \]
\[ = 1 \]

Remark 5.2.21: The geometric series converges if \( |r| < 1 \) and diverges if \( |r| \geq 1 \).
Exercise 5.2.22:
State whether each of the following series converges or diverges, and then find the sum of the series if it converges:

\[ \sum_{n=1}^{\infty} \frac{1}{2^n-1} \]

\[ \sum_{n=1}^{\infty} \frac{5}{3^n-1} \]

\[ \sum_{n=1}^{\infty} 3(2^{n-1}) \]

\[ \sum_{n=1}^{\infty} 7(-1)^{n-1} \]

### S 5.3: Power Series, Taylor Series and Maclaurian Series

**Definition 5.3.1:** A power series is a series of the form

\[ \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots \]

**Definition 5.3.2:** The Maclaurian series for a function \( F \) is

\[ f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \ldots + \frac{f^{(n)}(0)}{n!} x^n + \ldots \]

(i.e. \( f(x) = f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \ldots + \frac{f^{(n)}(0)}{n!} x^n + \ldots \) about \( x = 0 \)).

**Example 5.3.3:** Find the Maclaurian series for the function \( f(x) = e^x \).

**Solution:** Since \( f(x) = e^x \), \( f'(x) = e^x \), \( f''(x) = e^x \), \ldots,

\( f^{(n)}(x) = e^x \).
Then \( f(0) = e^0 = 1 \), \( f'(0) = e^0 = 1 \), \( f''(0) = e^0 = 1 \), \ldots, \( f^{(n)}(0) = e^0 = 1 \), and this implies that the Maclaurian series for the function 
\( f(x) = e^x \) is

\[
e^x = f(x) = f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots
\]

\[= 1 + 1 \cdot x + \frac{1}{2!} x^2 + \cdots + \frac{1}{n!} x^n + \cdots
\]

\[= \sum_{k=0}^{\infty} \frac{1}{k!} x^k.
\]

**Example 5.3.4**: Find the Maclaurian series for the function \( f(x) = \cos x \).

**Solution**: Since \( f(x) = \cos x \), \( f'(x) = -\sin x \),

\[
f''(x) = -\cos x \quad f^{(3)}(x) = \sin x \quad \vdots
\]

\[f^{(2k)}(x) = (-1)^k \cos x \quad f^{(2k+1)}(x) = (-1)^{k+1} \sin x
\]

Then \( f^{(2k)}(0) = (-1)^k \cos 0 = (-1)^k \cdot 1 = (-1)^k \)
and \( f^{(2k+1)}(0) = (-1)^{k+1} \sin 0 = (-1)^{k+1} \cdot 0 = 0 \), and this implies that the Maclaurian series for the function 
\( f(x) = \cos x \) is

\[
\cos x = f(x) = f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots
\]

\[= 1 + 1 \cdot x + \frac{1}{2!} x^2 + \cdots + \frac{1}{n!} x^n + \cdots
\]

\[= \sum_{k=0}^{\infty} \frac{1}{k!} x^k.
\]

**Exercise 5.3.5**: Find the Maclaurian series for the function 
\( f(x) = \sin x \).
Definition 5.3.6: The Taylor series for the function $f$ about $x = a$ is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

Remark 5.3.7: The Maclaurian series are Taylor series with $a = 0$.

Example 5.3.8: Find the Taylor series of $\cos x$ about $x = 2\pi$.

Solution:

Since $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f^{(3)}(x) = \sin x$, $f^{(4)}(x) = \cos x$, $f^{(5)}(x) = -\sin x$, $f^{(6)}(x) = -\cos x$, $f^{(7)}(x) = \sin x$, and so on.

Then $f^{(2k)}(2\pi) = (-1)^k \cos(2\pi) = (-1)^k \cdot 1 = (-1)^k$ and $f^{(2k+1)}(2\pi) = (-1)^{k+1} \sin(2\pi) = (-1)^{k+1} \cdot 0 = 0$ and this implies that the Taylor series of $f(x) = \cos x$ about $x = 2\pi$ is

$$\cos x = f(x) = f(2\pi) + f'(2\pi)(x-2\pi) + \frac{f''(2\pi)}{2!}(x-2\pi)^2 + \frac{f^{(3)}(2\pi)}{3!}(x-2\pi)^3 + \ldots$$

$$= \cos(2\pi) - \sin(2\pi)(x-2\pi) - \frac{\cos(2\pi)(x-2\pi)^2}{2!} + \frac{\sin(2\pi)(x-2\pi)^3}{3!} + \ldots$$

$$= 1 - 0(x-2\pi) - \frac{1}{2!}(x-2\pi)^2 + \frac{0}{3!}(x-2\pi)^3 + \frac{1}{4!}(x-2\pi)^4 + \ldots$$

$$= 1 - \frac{(x-2\pi)^2}{2!} + \frac{(x-2\pi)^4}{4!} + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (x-2\pi)^{2k}}{(2k)!}.$$
Example 5.3.9: Find the Taylor series for the function
\( f(x) = \frac{1}{x} \) about \( x = 1 \).

Solution:
Since \( f(x) = \frac{1}{x} = x^{-1} \), \( f'(x) = -1 \cdot x^{-2} = -\frac{1}{x^2} \),

\[
\begin{align*}
f''(x) &= 2x^{-3} = \frac{2}{x^3} , \quad f^{(3)}(x) = -6x^{-4} = \frac{-6}{x^4} = \frac{-3!}{x^4} \\
f^{(k)}(x) &= (-1)^k \cdot \frac{k!}{x^{k+1}} , \ldots
\end{align*}
\]

Then \( f^{(k)}(1) = (-1)^k \cdot \frac{k!}{(1)^{k+1}} = (-1)^k \cdot k! \) and this implies that the Taylor series of \( f(x) = \frac{1}{x} \) about \( x = 1 \) is

\[
\begin{align*}
\frac{1}{x} &= f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!} (x-1)^2 + \ldots + \frac{f^{(n)}(1)}{n!} (x-1)^n + \ldots \\
&= 1 + (-1)^1 (1!) (x-1) + \frac{(-1)^2 \cdot 2!}{2!} (x-1)^2 + \frac{(-1)^3 \cdot 3!}{3!} (x-1)^3 + \ldots \\
&= 1 - (x-1) + (x-1)^2 - \frac{1}{2} (x-1)^3 + \frac{1}{4} (x-1)^4 - \ldots \\
&= \sum_{k=0}^{\infty} (-1)^k (x-1)^k
\end{align*}
\]

Exercise 5.3.10: Find the Taylor series for the function
\( f(x) = \frac{1}{x} \) about \( x = -1 \).

(ans. \( \frac{1}{x} = f(x) = \sum_{k=0}^{\infty} (-1) \cdot (x+1)^k \)).
5.4: Fourier Series

Definition 5.4.1: The Fourier series of a function \( f(x) \) defined on the interval \(-L < x < L\) is

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] \quad (1)
\]

where

\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx
\]

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx
\]

Remarks 5.4.2:

Suppose that \( f \) is a function defined over the symmetric interval \(-L < x < L\). Assume that \( f \) is expressible as the trigonometric series given by equation (1). If \( m \) and \( n \) are positive integers, then

\[
(1) \quad \int_{-L}^{L} \cos \frac{n\pi x}{L} \, dx = 0
\]

\[
(2) \quad \int_{-L}^{L} \sin \frac{n\pi x}{L} \, dx = 0
\]
\[ \int_{-L}^{L} \cos \frac{n \pi x}{L} \cos \frac{m \pi x}{L} \, dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases} \]

\[ \int_{-L}^{L} \sin \frac{n \pi x}{L} \cos \frac{m \pi x}{L} \, dx = 0 \]

\[ \int_{-L}^{L} \sin \frac{n \pi x}{L} \sin \frac{m \pi x}{L} \, dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases} \]

**Example 5.4.3:** Find the Fourier series expansion of the function
\[ f(x) = \begin{cases} 1 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases} \]

**Solution:**

Since \( L = \pi \), then \( a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx \)

\[ = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \, dx \]

\[ = \frac{1}{\pi} \int_{-\pi}^{0} \, dx + \frac{1}{\pi} \int_{0}^{\pi} x \, dx \]

\[ = \frac{1}{\pi} \left[ x \right]_{-\pi}^{0} + \frac{1}{\pi} \left[ \frac{x^2}{2} \right]_{0}^{\pi} \]

\[ = \frac{1}{\pi} (-\pi) + \frac{1}{\pi} \left( \frac{\pi^2}{2} \right) = 1 + \frac{\pi}{2} \]
CH6: Hyperbolic Functions and Inverse Hyperbolic Functions

5.6.1: Hyperbolic Functions

Definition 5.6.1.1:

1. Hyperbolic cosine of x: \( \cosh x = \frac{e^x + e^{-x}}{2} \)

2. Hyperbolic sine of x: \( \sinh x = \frac{e^x - e^{-x}}{2} \)

3. Hyperbolic tangent: \( \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \)

4. Hyperbolic cotangent: \( \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \)

5. Hyperbolic secant: \( \text{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \)

6. Hyperbolic cosecant: \( \text{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \)
5.6.2: Inverse Hyperbolic Functions

The inverse hyperbolic sine function is defined by
\[ y = \sinh^{-1} x \quad \text{iff} \quad x = \sinh y \]

The inverse hyperbolic cosine function is defined by
\[ y = \cosh^{-1} x \quad \text{iff} \quad x = \cosh y \quad \text{and} \quad y \geq 0 \]

or by \( \{(x, y) \mid x = \cosh y, \ y \geq 0\} \)

Similarly, \( \tanh, \ coth \) and \( \csch \) have inverses, denoted by \( \tanh^{-1}, \ coth^{-1} \) and \( \csch^{-1} \).

Remark 6.2.1

\( \sech \) does not have a unique inverse.

We define the inverse hyperbolic secant by
\[ y = \sech^{-1} x \quad \text{iff} \quad x = \sech y \quad \text{and} \quad y \geq 0 \]

or \( \{(x, y) \mid x = \sech y, \ y \geq 0\} \).

Remarks 6.2.2

Since the natural logarithmic function is the inverse of the exponential function, then the inverse hyperbolic functions may be expressed in terms of \( \ln x \).
Let $y = \cosh^{-1} x$, where $x \geq 1$. Then

$$x = \cosh y = \frac{1}{2} (e^y + e^{-y}) \text{ for } y \geq 0.$$  

$$\Rightarrow 2x e^y = e^{2y} + 1$$  

$$\Rightarrow (e^y)^2 - 2x (e^y) + 1 = 0$$  

$$\Rightarrow e^y = x \pm \sqrt{x^2 - 1} \text{ or } y = \ln (x \pm \sqrt{x^2 - 1})$$  

$$\Rightarrow \cosh^{-1} x = \ln (x + \sqrt{x^2 - 1}) \quad [\text{since } \cosh^{-1} x \text{ is the larger of these two values of } y]$$

Similarly

$$\sinh^{-1} x = \ln (x + \sqrt{x^2 + 1}) \quad (\text{for any } x),$$  

$$\cosh^{-1} x = \ln (x + \sqrt{x^2 - 1}) \quad (x \geq 1),$$  

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x} \quad (-1 < x < 1).$$