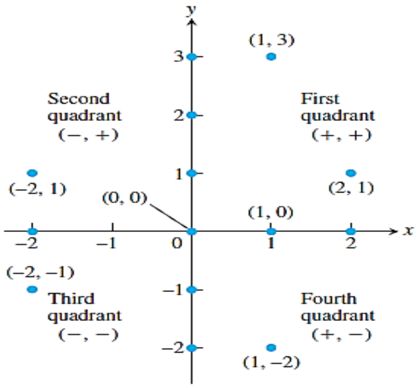
Chapter 1

The rate of change of function.

1.1 Cartesian coordinates in the plane

Points in the plane can be identified with ordered pairs of real number. To begin, we draw two perpendicular coordinate lines that intersect at the 0-point of each line.

These lines are called **Coordinate axes** in the plane



The Coordinat
or Cartesian p
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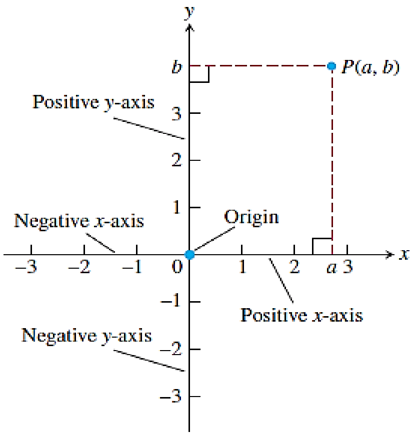


1.1.1 Increment and straight line:

When a particle moves from one point in the plane to another, the net changes in its coordinates are called **increments**. They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point. If x changes from x_1 to x_2 the increment in x is $\Delta x = x_2 - x_1$

EXAMPLE 1: A Particle moves from A to B in coordinate plane. Find the increment Δx and Δy in the particles coordinate.

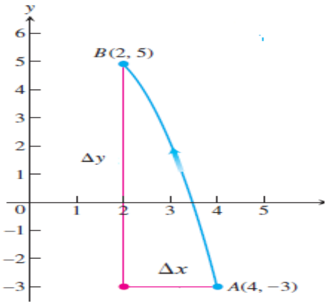
Sol: $\Delta x = x_2 - x_1$ $\qquad$ $y_2 = 5 \quad y_1 = -3$

$= 2 - 4$ $\qquad$ $x_2 = 2 \quad x_1 = 4$

$= -2$

$\Delta y = y_2 - y_1$

$= 5 - (-3) \; = 8$

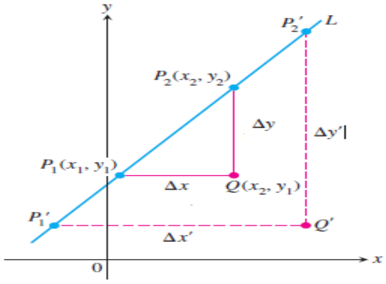


H.W 1. A Particle in the plane move from (-2, 5) to the y- axis in such a way that Δy is equaled $3\Delta x$ what were the Particle new coordinate?

2. The pressure P experienced by a diver under water is related to the diver's depth d by an equation of the form $\mathrm{P} = \mathrm{kd} + 1$ (k a constant). At the surface, the pressure is 1 atmosphere. The pressure at 100 meters is about 10.94 atmospheres. Find pressure at 50 m.

1.2 The slop of straight line:

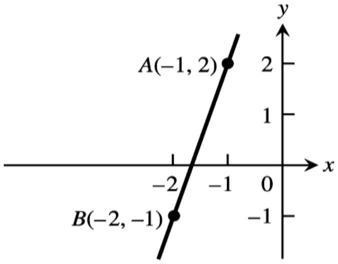
Any nonvertical line in the plane has the property that the ratio $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ has the same value for every choice of the two points and on the line.



EXAMPLE 1 Plot the points and find the slope (if any) of the line they determine.

1) A (-1 , 2) B (-2 , -1)

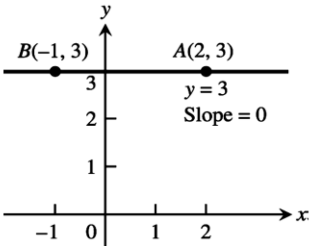
2) A (2 , 3) B (-1 , 3)



Sol 1)

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-1-2}{-2+1} = \frac{-1}{-1} = 3$$

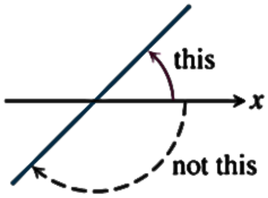


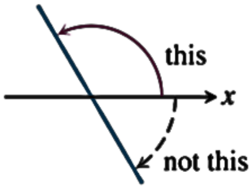
2) $$m = \frac{\Delta y}{\Delta x}$$

$$m = \frac{3-3}{-1-2} = \frac{0}{-3} = 0$$

1.2 The angle of inclination

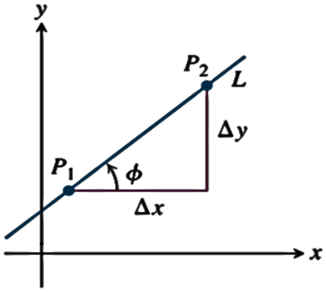
The angle of inclination of a line that crosses the *x*-axis is the smallest counterclockwise angle from the *x*-axis to the line.





The relationship between the slope m of a nonvertical line and the line's angle of inclination ϕ is shown in Figure.

$$m = \frac{\Delta y}{\Delta x} = \tan\phi$$



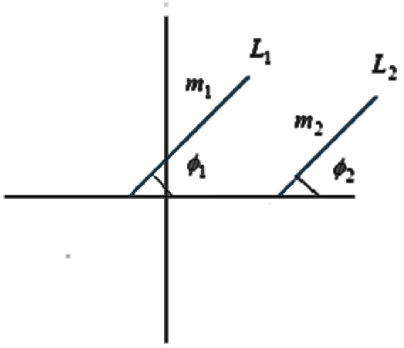
EXAMPLE 1 Find the slop of the line that make angle 60 with x- axis ?

Sol
$$\begin{aligned} m &= \tan\phi \\ &= \tan 60 \\ &= \sqrt{3} \end{aligned}$$

1.3.1 Parallel and perpendicular line

a) Lines that are parallel have equal angles of inclination, so they have the same slope (if they are not vertical)

$$\mathrm{L}_1 // \mathrm{L}_2 \quad , \quad \phi_1 = \phi_2$$
$$m_1 = m_2$$



Hint: 1. The slope of vertical line is undefined $\Delta x = 0$

2. The slope of horizontal line is equal zero $\Delta y = 0$

b) If two nonvertical lines and are perpendicular, their slopes m_1 and m_2 satisfy

$$m_1 m_2 = -1$$

$$m_1 = \frac{-1}{m_2} \quad , \quad m_2 = \frac{-1}{m_1}$$

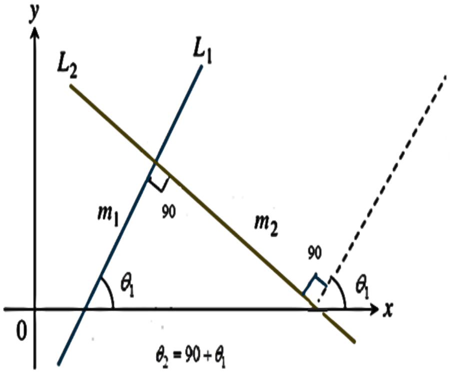
➡ $$m_2 = \tan\theta_2 = \frac{\sin\theta_2}{\cos\theta_2}$$

$$\sin(90+\theta_1) = \cos\theta_1$$

$$\cos(90+\theta_1) = -\sin\theta_1$$

$$\tan\theta_2 = \frac{\cos\theta_1}{-\sin\theta_1}$$

$$\tan\theta_2 = \frac{-1}{\tan\theta_1}$$



$$m_2 = \frac{-1}{m_1}$$

1.4 The slope form general equation

$$Ax + By = c$$

$$m = \frac{-A}{B}$$

EXAMPLE 1 Find the slope of the line $2y = 3x + 4$

Sol

$$2y = 3x + 4$$ ➡ $$A = -3 \;,\; B = 2$$

$$-3x + 2y = 4$$

$$m = \frac{-A}{B}$$

$$m = \frac{-(-3)}{2} = \frac{3}{2}$$

1.4 Equation of straight line:

1.4.1 Equation of nonvertical straight line:

a) Point –slope equation

We can write an equation for a nonvertical straight line *L* if we know its slope *m* and the coordinates of one point on it.

$$m = \frac{y - y_1}{x - x_1}$$

$$y - y_1 = m\,(x - x_1)$$

$$\boxed{y = y_1 + m\,(x - x_1)}$$

The equation is the point - slope equation of the line that passes through the point (x_1, y_1) and has slope m.

b) Slope intercept equation

A line with slope m and y-intercept b passes through the point (0, b), so it has equation.

$$y = b + m\,(x - 0)$$

Or $\boxed{y = mx + b}$

The equation is called the **slope-intercept equation** of the line with slope m and y-intercept b.

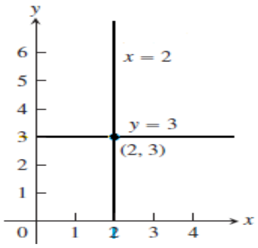
C) General linear equation

$$\boxed{Ax + By = c}$$

The equation is called the **general linear equation** in x and y because its graph always represents a line and every line has an equation in this form (including lines with undefined slope).

1.4.2 Equation of vertical and horizontal lines

All points on the vertical line through the point a on the x-axis have x-coordinates equal to a $\longrightarrow$ $x = a$ (vertical line) similarly $y = b$, (horizontal line)



Home work (H.W)

12) Find the equation of line that make angle 30° with y-axis and passes through (1, 2)

13) Find an equation of line for

1) Line through P (2, 1) and parallel to $L \quad y = x + 2$

2) Find an equation for the line through
point P and perpendicular to L

14) Find the slope of the line $\frac{x}{2} + \frac{y}{3} = 1$ and what are the line x-and y-intercept?

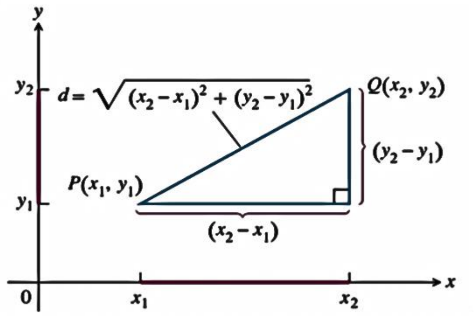
15) Find the equation of line that passes through P (2, -1) and parallel to the line $2x - 7y = 1$.

16) Find the equation of line that passes through P (1,4) and perpendicular on the $4x + 6y = 7$?

1.5 Distance between two points

The distance between points in the plane is calculated with a formula that comes from relation.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



EXAMPLE 1: Find distance between the point (1, -3) and (4, 2)

Sol

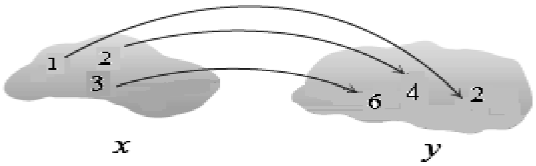
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(4-1)^2 + (2-(-3))^2}$$
$$= \sqrt{9 + 25}$$
$$= \sqrt{34}$$

Function and graphs

The values of one variable quantity which we might call y, depend on the values of another variable quantity which we might call x

If the value of y is completely determined by the value of x, so y is function of x i.e.

$$y = f(x)$$



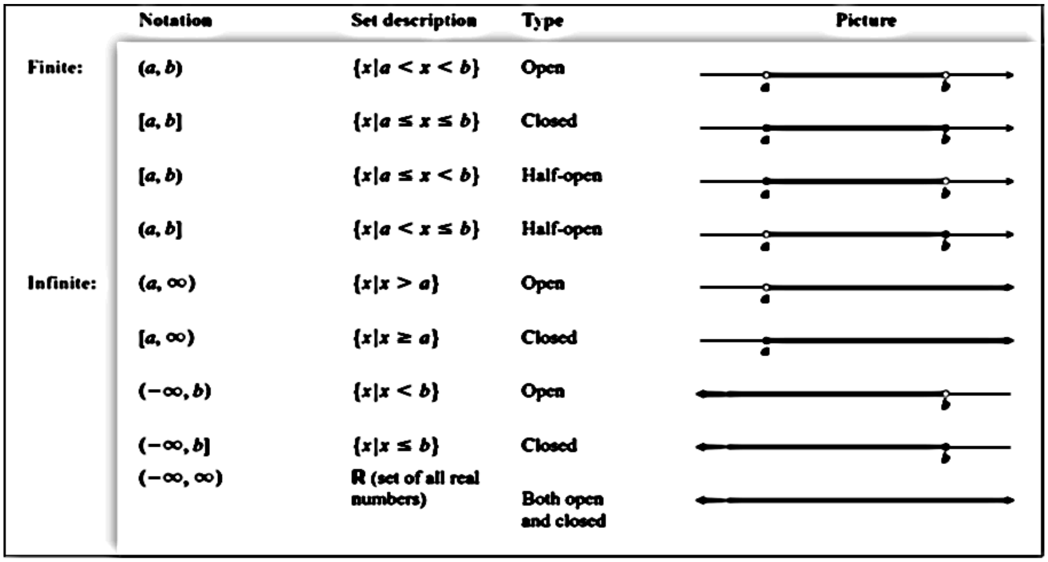
$$D_f \ \{1,2,3\}$$
$$R_f \ \{2,4,6\}$$

$$\begin{aligned} f(1) &= 2 \\ f(2) &= 4 \quad \longrightarrow \quad f(x) = 2x \\ f(3) &= 6 \end{aligned}$$

Note The variable x called independent variable, the variable y called dependent variable on value of x

Function: A function from a set D to a set y is a rule that assigns a unique (single) element $f(x) \in y$ to each element $x \in D$.

1. The set D of all possible input values is call "Domain"
2. The set of all values of $f(x)$ as x varies throughout D is called Range of the



	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
	$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

function. The domains D_f and range R_f of many functions in mathematics are interval of real number are shown in figure.

Root function

General form $y = \sqrt{f(x)} + k$

$D_f \longrightarrow f(x) \geq 0$

R_f we have two cases

1. If the value of D_f unbounded

   $R_f = \{y: \ y \geq k\}$

Find domain D_f and range R_f of

EXAMPLE 1: $y = \sqrt{x}$

Sol

$x \geq 0$

$D_f = \{x : x \geq 0\}$

$R_f = \{y : y \geq 0\}$

H.W

Ex 2: $y = \sqrt{4 - x}$

Ex 3: $y = \sqrt{1 - 3x}$

Ex 4: $f(x) = \sqrt{2x - 3} + 7$

2. If the value of D_f bounded $(-a \leq x \leq a)$

   $R_f = (0 \leq y \leq a)$

Relative function

General form $y = \dfrac{f(x)}{g(x)}$

$D_f = R / \{g(x) = 0\}$

To find R_f rewrite function ($x \to f(y)$)

Find domain D_f and range R_f

EXAMPLE 1: $y = \dfrac{1}{x}$

$D_f = R/\{x=0\}$

$R_f \rightarrow \quad y=\frac{1}{x} \qquad x=\frac{1}{y} \qquad R_f = R/\{y=0\}$

H.W **Ex 1**: $y=\frac{x}{x-1}$ **Ex 2**: $y=\frac{3x-3}{1-7x}$ **Ex 3**: $y=\frac{2x-4}{x+3}$

Ex 4: $y=\sqrt{\frac{1+x}{1-x}}$ **Ex 5**: $f(x)=1+x^2$ **Ex 6**: $f(t)=\frac{1}{\sqrt{t}}$

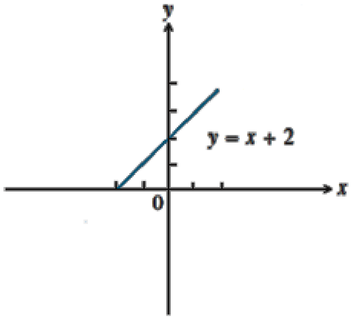
Graphs of functions

Another way to visualize a function is its graph. If f is a function with domain D, its graph consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f.

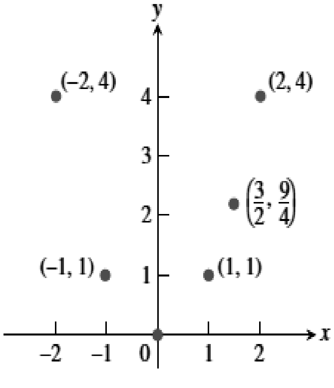
EXAMPLE 1: Graph the function over the interval $-2 \le x \le 2$

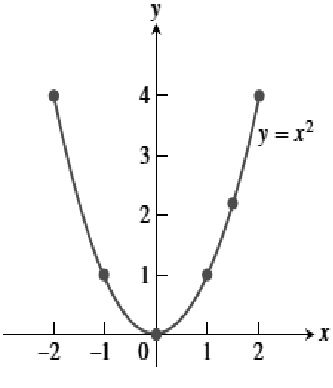
1. $y = x+2$

x	y
2	4
1	3
0	2
-1	1
-2	0



2. $y = x^2$





H.W **Ex 2**:Graph the function over the interval $-2 \le x \le 2$

1. $y = x^2 + 1$ 2. $y = x^2 - 1$

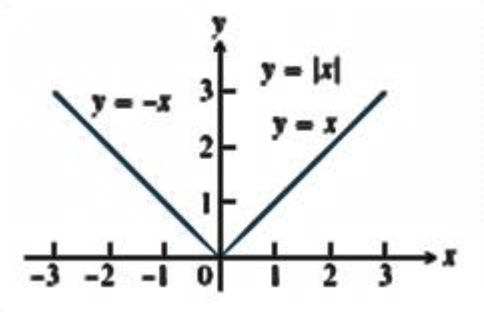
3. $y = \sqrt{1 - x^2}$ 4. $y = -\sqrt{1 - x^2}$

Graphing piecewise –Defined function

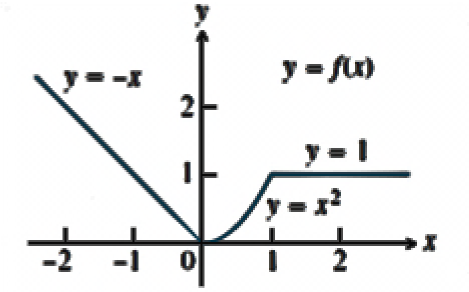
Sometimes a function is described by using different formulas on different part of its domain:

EXAMPLE 1: Graph the absolute value function

Sol $$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



EXAMPLE 2: Graph the function $f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$



H.W Graph the functions $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 < x \leq 2 \end{cases}$

10: Limits and continuity

Let $f(x)$ be defined on an open interval about x , except possibly at x itself. The limit of $f(x)$ as approaches x is the number L $\lim_{x \to x} f(x) = L$.

The limit of a function $f(x)$ as $x \to x$ never depend on what happen when $x = x$

Right hand limit $\lim\limits_{x \to x^+} f(x) = L$

Left hand limit $\lim\limits_{x \to x^-} f(x) = L$

A function $f(x)$ has a limit at point x if and only if the right and left hand limit at x exit and equal

$$\lim_{x \to x^+} f(x) = L \Leftrightarrow \lim_{x \to x^-} f(x) = L \quad \text{and} \lim_{x \to x} f(x) = L$$

10.1 The Limit Laws:

If L, M, C and K are real number and $\lim\limits_{x \to c} f(x) = L$, $\lim\limits_{x \to c} g(x) = M$

1) Sum rule $\lim\limits_{x \to c}(f(x) + g(x)) = L + M$
2) Deference $\lim\limits_{x \to c}(f(x) - g(x)) = L - M$
3) Product $\lim\limits_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
4) Constant multiple $\lim\limits_{x \to c}(k \cdot f(x)) = k \cdot L$
5) Quotient rule $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \neq 0$
6) power rule $\lim\limits_{x \to c}(f(x))^{r/s} = L^{r/s}$

EXAMPLE 1: Find the limit of the function $f(x) = x^3 + 4x^2 - 3$ at $x \to c$

Sol:

$$f(x) = x^3 + 4x^2 - 3$$

$$\lim_{x \to c}(x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$$

$$= c^3 + 4c^2 - 3$$

H.W **Ex 1**: $\lim\limits_{x \to c} \dfrac{x^4 - x^2 - 1}{x^2 + 5}$

H.W **Ex 2:** $\lim\limits_{x\to 2}\sqrt{4x^2-3}$

Find the limit of function

a) The limit laws

$$\lim_{x\to 2}(4) \quad , \quad \lim_{x\to 2}(5x-3) \quad , \quad \lim_{x\to -2}\frac{3x+4}{x+5}$$

$\lim\limits_{x\to 1}(x^2-2x+3)$ (To finding limits by calculating)

$$\lim_{x\to 3}\sqrt{x^2+3x-4} \quad , \quad \lim_{x\to -3}\frac{x^2+x-1}{2x-4x^2} \quad , \quad \lim_{x\to 13}(4)$$

b) Limit of Rational Function

EXAMPLE 1: $\lim\limits_{x\to 1}\frac{x^2-1}{x-1}=\lim\limits_{x\to 1}\frac{(x-1)(x+1)}{(x-1)}=\lim\limits_{x\to 1}(x+1)=1+1=2$

Ex 2: $\lim\limits_{x\to 2}\frac{x-2}{x^2-4}$

Ex 3: $\lim\limits_{x\to -3}\frac{x^2-9}{x+3}$

Ex 4: $\lim\limits_{x\to 2}\frac{x^4-16}{x-2}$

Ex 5: $\lim\limits_{x\to 1}\frac{x-1}{\sqrt{x}-1}$

➡ Limit of Rational function can be found by substitution if the limit of denominator is not zero

Hint $a^3-b^3=(a-b)(a^2+ab+b^2)$

$a^3+b^3=(a+b)(a^2-ab+b^2)$

Ex 1 $(x^3-8)=(x^3-2^3)=(x-2)(x^2+2x+4)$

c) Limit at infinity of Rational function

If the value of $x\to\pm\infty$

EXAMPLE 1: $\lim\limits_{x\to\infty}\frac{x^3-2x+1}{2+x^2-4x^3}$

Sol:

$$\lim_{x\to\infty}\frac{\frac{x^3}{x^3}-\frac{2x}{x^3}+\frac{1}{x^3}}{\frac{2}{x^3}+\frac{x^2}{x^3}-\frac{4x^3}{x^3}}$$

$$\lim_{x\to\infty}\frac{1-\frac{2}{x^2}+\frac{1}{x^3}}{\frac{2}{x^3}+\frac{1}{x}-4}=\frac{1-\frac{2}{\infty}+\frac{1}{\infty}}{\frac{2}{\infty}+\frac{1}{\infty}-4}=\frac{1-0+0}{0+0-4}$$

$$\lim_{x\to\infty}\frac{x^3-2x+1}{2+x^2-4x^3}=-\frac{1}{4}$$

Ex 2: $\lim_{x\to\infty}\frac{4x^5+2x^3+3x-1}{2x^7-3x^4+2x-7}$ ans (0)

Ex 3: $\lim_{x\to\infty}\frac{2x-3}{x+\sqrt{x^2-1}}$ ans (1)

Ex 4: $\lim_{x\to\infty}\frac{7x+2}{3x+\sqrt{x^2+1}}$ ans (7/4)

d) Limit of Root function

If we have Root ± number

Root ± Root

EXAMPLE 1: Find limit of $\frac{1-\sqrt{x+1}}{x}$ at $x\to 0$

Sol:

$$\lim_{x\to 0}\frac{1-\sqrt{x+1}}{x}\cdot\frac{1+\sqrt{x+1}}{1+\sqrt{x+1}}=\lim_{x\to 0}\frac{1-(x+1)}{x+x\sqrt{x+1}}$$

$$=\lim_{x\to 0}\frac{1-x-1}{x(1+\sqrt{x+1})}=\lim_{x\to 0}\frac{-x}{x(1+\sqrt{x+1})}=\frac{-1}{(1+\sqrt{x+1})}=\frac{-1}{(1+\sqrt{0+1})}$$

$$=\frac{-1}{(1+1)}=-\frac{1}{2}$$

H.W **Ex 2** $\lim_{n\to\infty}\sqrt{n^2+1}-n$ **Ex 3:** $\lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2}$ **Ex 4:** $\lim_{x\to 0}\frac{x^3-a^3}{x^4-a^4}$

11. Continuity

Continuous function: A function is continuous if it is continuous at each point of its domain.

The Continuity test

The function $y = f(x)$ is continuous at $x = c$ if and only if all three of following statement are true

1. $f(c)$ exit "c in the domain of f"
2. $\lim\limits_{x \to c} f(x)$ exit "f has a limit at $x \to c$"
3. $\lim\limits_{x \to c} f(x) = f(c)$ "The limit equal the function value"

Hint if f continues at $x \to c$ and g continuous at $x \to c$

1. $f \cdot g$
2. $g \cdot f$
3. $k \cdot g$
4. f / g

.... continuous

EXAMPLE 1: Determine if the following function is continuous at $x = 1$?

$$f(x) = \begin{cases} 3x - 5 & \text{at } x \neq 1 \\ 2 & \text{at } x = 1 \end{cases}$$

Sol:

1) $f(1) = 2$
2) $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1}(3x - 5) = (3(1) - 5) = -2$

$$f(1) \neq \lim_{x \to 1} f(x)$$

The function $f(x)$ is not continuous at $x = 1$

H.W **Ex 2:** Determine if the $f(x)$ is continuous at $x = 3$?

$$f(x) = \begin{cases} x^2 - 1 & \text{at } x = 3 \\ x^2 + 2 & \text{at } x \neq 3 \end{cases}$$

H.W **Ex 3:** if $f(x)=\dfrac{x+3}{x^2-1}$ where is $f(x)$ continuous, and where it is discontinuous ?

limit of trigonometric function

1. $\lim\limits_{x\to 0}\sin x=0$ $\quad\quad\quad\quad$ $\sin(0)=0$
2. $\lim\limits_{x\to 0}\cos x=1$ $\quad\quad\quad\quad$ $\cos(0)=1$
3. $\lim\limits_{x\to 0}\tan x=0$ $\quad\quad\quad\quad$ $\tan(0)=0$
4. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$
5. $\lim\limits_{x\to 0}\dfrac{\sin ax}{ax}=\lim\limits_{x\to 0}\dfrac{ax}{\sin ax}=1$
6. $\lim\limits_{x\to 0}\dfrac{\tan ax}{ax}=\lim\limits_{x\to 0}\dfrac{ax}{\tan ax}=1$
7. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}=0$

EXAMPLE 1: Prove $\lim\limits_{x\to 0}\sin x=0$

Sol: $\lim\limits_{x\to 0}\sin x=\sin(0)=0$

EXAMPLE 2: Prove $\lim\limits_{x\to 0}\cos x=1$

Sol: $\lim\limits_{x\to 0}\cos x=\cos(0)=1$

EXAMPLE 3: Prove $\lim\limits_{x\to 0}\tan x=0$

Sol: $\lim\limits_{x\to 0}\tan x=\tan(0)=0$

EXAMPLE 4: Prove $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

Sol: the Taylor series for $\sin x$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\cdots$$

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\left(\frac{x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\cdots\cdots}{x}\right)$$

at $x\to 0$ $\sin x\approx x$ so we can neglected $\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\cdots$

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{x}{x}$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Exercises

Q1) Find the limits

a) $=\lim_{x\to -1}\frac{3x^2}{2x-1}$

b) $=\lim_{x\to \pi/2} x\sin x$

c) $=\lim_{x\to \pi}\frac{\cos x}{1-\pi}$

Q2) Calculate limits using the limit laws

a) $=\lim_{t\to 1}\frac{t^2+3t+2}{t^2-t-2}$

b) $=\lim_{x\to -1}\frac{\sqrt{x^2+8}-3}{x+1}$

c) $=\lim_{u\to 1}\frac{u^4-1}{u^3-1}$

d) $=\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}$

e) $=\lim_{x\to 4}\frac{4x-x^2}{2-\sqrt{x}}$

Q3) Using $=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$

Show that a) $=\lim_{h\to 0}\frac{\cos h-1}{h}=0$

b) $= \lim_{x\to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$

Transcendental function

1.1 Logarithm function

- **Definition** $\log_a x$
- **Properties of Logarithm function**
- **Rule of Logarithm function**
- **Example**

1.2 Exponential function

- **Definition of**
- **Properties and rule of Exponential function**
- **Example**

1.3 Invers function

- **Example**

1.1 Logarithm function

Logarithms with Base a

Definition $\log_a x$

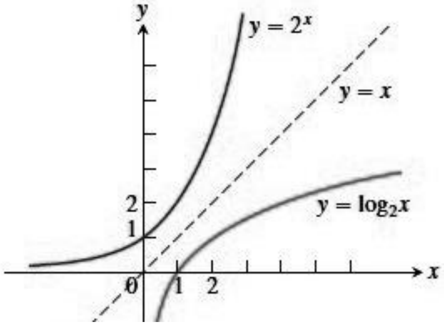
For any positive number $a \neq 1$ $\log_a x$ is the inverse function of a^x .

Example: $y = \log_2 x$ reflecting the graph of $y = a^x$ when $a = 2$ as shown in Fig.

So that mean

$$y = \log_a x$$

$$a^y = x$$



Inverse Equations for a^x and $\log_a x$

1. $a^{\log_a x} = x$ $\qquad x > 0$
2. $\log_a(a^x) = x$ $\qquad$ all x

Rules for base a logarithms for any numbers $x > 0$ and $y > 0$

1. Product Rule: $\log_a xy = \log_a x + \log_a y$

2. Quotient Rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$

3. Reciprocal Rule: $\log_a \frac{1}{y} = -\log_a y$

4. Power Rule: $\log_a x^y = y \log_a x$

Also

$$\log_a a = 1 \quad , \quad \log_a 1 = 0 \quad , \quad \log_a x = \frac{\ln x}{\ln a} \quad , \quad \log_a x = \frac{\log x}{\log a}$$

EXAMPLE 1: Prove that $\log_a x = \frac{\ln x}{\ln a}$

Proof: $a^{\log_a x} = x$

tak ln $\Rightarrow \ln a^{\log_a x} = \ln x$ using properties

$$\log_a x \cdot \ln a = \ln x \qquad \log_a x = \frac{\ln x}{\ln a}$$

EXAMPLE 2: Calculate $\frac{1}{\log_{10} 30} + \frac{1}{\log_3 30}$

Sol:

$$\frac{1}{\frac{\log 30}{\log 10}} + \frac{1}{\frac{\log 30}{\log 3}} \quad or \quad \frac{1}{\frac{\ln 30}{\ln 10}} + \frac{1}{\frac{\ln 30}{\ln 3}}$$

$$\Rightarrow \frac{\ln 10}{\ln 30} + \frac{\ln 3}{\ln 30} = \frac{\ln 10 + \ln 3}{\ln 30}$$

$$= \frac{\ln(10 \times 3)}{\ln 30} = \frac{\ln 30}{\ln 30} = 1$$

H.W **:** Find value of y

1. $y = 5^{\log_5 7} \quad \Rightarrow \quad y = 7$
2. $y = \log_6 36 \quad \Rightarrow \quad y = 2$
3. $y = \log_3(1/9) \quad \Rightarrow \quad y = -2$
4. $y = \log_3 \sqrt{3} \quad \Rightarrow \quad y = 1/2$
5. $y = \log_4(4^{2/3}) \quad \Rightarrow \quad y = 2/3$
6. $y = \log_x(1/\sqrt{x}) \quad \Rightarrow \quad y = -1/2$

Exponential function

$$y = a^x \Rightarrow a : \text{constant}$$
$$y = e^x \Rightarrow e \approx 2.718$$

Properties and rule of Exponential function

1. $e^x \cdot e^y = e^{x+y}$
2. $e^x / e^y = e^{x-y}$
3. $(e^x)^n = e^{nx}$

Rule

1. $\ln e = 1$
2. $\ln 1 = 0$
3. $\ln e^u = u \ln e = u$
4. $e^{\ln 1} = e^0 = 1$
5. $e^{\ln u} = u$

EXAMPLE 1: Solve for $e^{\ln y + 2x} = e^{\ln(x+1)}$

Sol:

$$e^{\ln y + 2x} = e^{\ln(x+1)}$$
$$e^{\ln y} \cdot e^{2x} = (x+1)$$
$$y \cdot e^{2x} = (x+1)$$
$$y = (x+1)/e^{2x}$$

1.2 Invers function

IF $y = \sin^{-1} x \Rightarrow x = \sin y$ OR $x = \sin y \Rightarrow y = \sin^{-1} x$

EXAMPLE 1: Prove that $\sin^{-1}(-x) = -\sin^{-1} x$

Sol:

Let $y = \sin^{-1}(-x) \longrightarrow -x = \sin y \rightarrow x = -\sin y \longrightarrow y = -\sin^{-1}(x)$

Hyperbolic Function

Definition of hyperbolic function

1. $\sinh x = \frac{1}{2}(e^x - e^{-x}) = \frac{(e^x - e^{-x})}{2}$

2. $\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{(e^x + e^{-x})}{2}$

3. $\tanh x = \frac{\sinh x}{\cosh x} = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$

4. $\coth x = \frac{\cosh x}{\sinh x} = \frac{(e^x + e^{-x})}{(e^x - e^{-x})}$

5. $\mathrm{sechx} = \frac{1}{\cosh x} = \frac{2}{(e^x + e^{-x})}$

6. $\mathrm{cschx} = \frac{1}{\sinh x} = \frac{2}{(e^x - e^{-x})}$

Identities

1. $\cosh^2 x - \sinh^2 x = 1$ Prove?
2. $\sinh 2x = 2\sinh x \cosh x$ Prove?
3. $\cosh 2x = \cosh^2 x + \sinh^2 x$ Prove?
4. $\cosh^2 x = \frac{\cosh 2x + 1}{2}$ Prove?
5. $\sinh^2 x = \frac{\cosh 2x - 1}{2}$ Prove?
6. $\tanh^2 x = 1 - \mathrm{sech}^2 x$ Prove?
7. $\cosh^2 x = 1 + \mathrm{csch}^2 x$ Prove?
8. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
9. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
10. $\sinh x + \cosh x = e^x$
11. $\cosh 2x = 2\sinh^2 x + 1$
12. $\cosh 2x = 2\cosh^2 x - 1$
13. $\cosh(-x) = \cosh x$

14. $\sinh(-x) = -\sinh x$

15. $\cosh x - \sinh x = e^{-x}$

EXAMPLE 1: Prove that $\sinh x + \cosh x = e^{x}$

Sol:

$$\sinh x = \frac{(e^{x} - e^{-x})}{2} \qquad \cosh x = \frac{(e^{x} + e^{-x})}{2}$$

$$\text{L.H.S } \frac{(e^{x} - e^{-x})}{2} + \frac{(e^{x} + e^{-x})}{2} = \frac{e^{x}}{2} - \frac{e^{-x}}{2} + \frac{e^{x}}{2} + \frac{e^{-x}}{2}$$

$$= \frac{e^{x}}{2} + \frac{e^{x}}{2} = e^{x} \text{ R.H.S}$$

EXAMPLE 2: Prove that $\cosh x - \sinh x = e^{-x}$

Sol

$$\text{L.H.S } \frac{(e^{x} + e^{-x})}{2} - \frac{(e^{x} - e^{-x})}{2} = \frac{e^{x}}{2} + \frac{e^{-x}}{2} - \frac{e^{x}}{2} + \frac{e^{-x}}{2}$$

$$= \frac{e^{-x}}{2} + \frac{e^{-x}}{2} = e^{-x} \text{ R.H.S}$$

Derivatives

Rule of Derivatives: Let c and n are constant, u, v and ware differentiable function of x:

1. $\frac{d}{dx} c = 0$
2. $\frac{d}{dx} u^{n} = n u^{n-1} \frac{du}{dx}$
3. $\frac{d}{dx}(\frac{1}{u}) = -\frac{1}{u^{2}} \frac{du}{dx}$
4. $\frac{d}{dx} cu = c \frac{du}{dx}$

5. $\frac{d}{dx}(u\cdot v)=u\cdot\frac{dv}{dx}+v\cdot\frac{du}{dx}$ and $\frac{d}{dx}(u\cdot v\cdot w)=u\cdot v\frac{dw}{dx}+u\cdot w\frac{dv}{dx}+v\cdot w\frac{du}{dx}$

6. $\frac{d}{dx}(\frac{u}{v})=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ where $v\neq 0$

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $y=\frac{x^2-1}{x^2+x-2}$

Sol:

$$y'=\frac{2x(x^2+x-2)-(2x+1)(x^2-1)}{(x^2+x-2)^2}$$

$$y'=\frac{2x^3+2x^2-4x-2x^3+2x-x^2+1}{(x^2+x-2)^2}=\frac{x^2-2x+1}{(x^2+x-2)^2}$$

H.W **Ex 2:** $y=\frac{12}{x}-\frac{4}{x^2}+\frac{3}{x^4}$

H.W **Ex 3**: $y=(2x^3-3x^2+6x)^{-5}$

H.W **Ex 4**: $y=\frac{x^2-1}{x^2+x-2}$

The chain rule

1. Suppose that $h=g\cdot f$ is the composite of the differentiable functions $y=g(t)$ and $x=f(t)$, then h is a differentiable function of x whose derivative at each value of x is

$$\boxed{\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}}$$

EXAMPLE 1: Find $\frac{dy}{dx}$ if $y=\frac{1}{t^2+1}$, $x=\sqrt{4t+1}$

Sol:

$$y=(t^2+1)^{-1},\ x=\sqrt{4t+1}$$

$$\frac{dy}{dx}=\frac{dy}{dt}/\frac{dx}{dt}=\frac{\frac{d}{dt}(t^2+1)^{-1}}{\frac{d}{dt}(4t+1)^{1/2}}$$

$$=\frac{-(t^2+1)^{-2}(2t)}{\frac{1}{2}(4t+1)^{-1/2}\cdot 4}=\frac{-2t\ (t^2+1)^{-2}}{2(4t+1)^{-1/2}}$$

$$=\frac{-t\ (t^2+1)^{-2}}{(4t+1)^{-1/2}}$$

2. If y is a differentiable function of t and t is a differentiable function of x ,then y is a differentiable of x:

   $y=g(t)$ and $t=f(x)$

$$\boxed{\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}}$$

EXAMPLE 1: Use the chain rule to express $\frac{dy}{dx}$ in terms of x and y

$$y=\frac{t^2}{t^2+1} \qquad , \qquad t=\sqrt{2x+1}=(2x+1)^{1/2}$$

Sol:

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{(t^2+1)2t-t^2(2t)}{(t^2+1)^2}\cdot\frac{1}{2}(2x+1)^{-1/2}(2)$$

$$=\frac{2t^3+2t-2t^3}{(t^2+1)^2}\cdot(2x+1)^{-1/2}$$

$$=\frac{2t}{(t^2+1)^2}\cdot\frac{1}{\sqrt{2x+1}} \qquad \text{sub } t$$

$$=\frac{2\sqrt{2x+1}}{(2x+1+1)^2}\cdot\frac{1}{\sqrt{2x+1}}=\frac{2}{(2x+2)^2}$$

Higher derivative

If a function $y=f(x)$ possesses a derivative at every point of some interval. We may form the function $f'(x)$ and take about its derivate if it has one.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx} f'(x)$$

This derivative is called the second derivative of y with respect to x. In some manner we may define third and higher derivatives using similar notations.

EXAMPLE 1: Find all derivatives of the following function.

$$y = 3x^3 - 4x^2 + 7x + 10$$

Sol:

$$y' = 9x^2 - 8x + 7$$

$$y'' = 18x - 8$$

$$y''' = 18$$

$$y'''' = 0$$

Implicit derivative

If the formula of f is an algebraic combination of power of x and y .To calculate the derivative of the implicitly defined functions. We simply differentiable both sides of the defining equation with respect to x.

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $x^2y^2 = x^2 + y^2$

Sol:

$$x^2 2yy' + 2xy^2 = 2x + 2y\,y'$$

$$x^2 2yy' - 2y\,y' = 2x - 2xy^2$$

$$y'(2x^2y - 2y) = 2x - 2xy^2$$

$$y' = \frac{2x - 2xy^2}{2x^2y - 2y} = \frac{x - xy^2}{x^2y - y}$$

Trigonometric function

1. $\sin u$	$\frac{d}{x}\sin u = \cos u \frac{du}{dx}$
2. $\cos u$	$\frac{d}{x}\cos u = -\sin u \frac{du}{dx}$
3. $\tan u$	$\frac{d}{x}\tan u = \sec^2 u \frac{du}{dx}$
4. $\cot u$	$\frac{d}{x}\cot u = -\csc^2 u \frac{du}{dx}$
5. $\sec u$	$\frac{d}{x}\sec u = \sec u \tan u \frac{du}{dx}$
6. $\csc u$	$\frac{d}{x}\csc u = -\csc u \cot u \frac{du}{dx}$

EXAMPLE 1: Prove that $\frac{d}{x}\tan u = \sec^2 u \frac{du}{dx}$

Sol:

$$\frac{d}{x}\tan u = \frac{d}{x}\frac{\sin u}{\cos u}$$

$$= \frac{\cos u \cos u - \sin u(-\sin u)}{\cos^2 u}\frac{du}{dx}$$

$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u}\frac{du}{dx}$$

$$= \frac{1}{\cos^2 u}\frac{du}{dx} = \sec^2 u \frac{du}{dx}$$

Hint:

1. $y = \sin^n u$	$y' = n\sin^{n-1} u \cos u \frac{du}{dx}$
2. $y = \cos^n u$	$y' = n\cos^{n-1} u(-\sin u)\frac{du}{dx}$
3. $y = \tan^n u$	$y' = n\tan^{n-1} u \sec^2 u \frac{du}{dx}$

4. $y=\cot^n u$ $\qquad y'=n\cot^{n-1}u(-\csc^2 u)\dfrac{du}{dx}$

5. $y=\sec^n u$ $\qquad y'=n\sec^{n-1}u\,(\sec u\tan u)\ \dfrac{du}{dx}$

6. $y=\csc^n u$ $\qquad y'=n\csc^{n-1}u\,(-\csc u\cot u)\dfrac{du}{dx}$

EXAMPLE 1: Find $\dfrac{dy}{dx}$ for the following function.

1. $y=\tan^2(\cos x)$

$$y'=2\tan(\cos x)\sec^2(\cos x)(-\sin x)$$

$$y'=-2\sin x\tan(\cos x)\sec^2(\cos x)$$

$$y=\sec^4 x-\tan^4 x$$

$$y'=4\sec^3 x\sec x\tan x-4\tan^3 x\sec^2 x$$

$$=4\sec^4 x\ \tan x-4\tan^3 x\sec^2 x$$

$$=\sec^2 x(4\sec^2 x\ \tan x-4\tan^3 x)$$

Transcendental function derivative

1- Logarithm function الدالة الوغارتيمية

(1) If $y=\ln x$ $\qquad \boxed{\Rightarrow\ \dfrac{d}{dx}\ln u=\dfrac{1}{u}\dfrac{du}{dx}}$

EXAMPLE: $y=\ln x$

$$y'=\frac{1}{x}$$

(2) $\boxed{\dfrac{d}{dx}\log_a u=\dfrac{d}{dx}(\dfrac{\ln u}{\ln a})=\dfrac{1}{\ln a}\dfrac{1}{u}\dfrac{du}{dx}}$

EXAMPLE: $y=\log_a x=\dfrac{\ln x}{\ln a}$ $\qquad \Rightarrow\ y'=\dfrac{1}{\ln a}\dfrac{1}{x}$

(3) If $y=\log x=\dfrac{\ln x}{\ln 10}$ $\qquad \Rightarrow y'=\dfrac{1}{\ln 10}\dfrac{1}{x}$

$$\boxed{\frac{d}{dx}\log u = \frac{1}{u}\frac{1}{\ln 10}\frac{du}{dx}}$$

EXAMPLE 1: $y = \ln(\sin x - \sec x)$

$$y' = \frac{\cos x - \sec x \tan x}{\sin x - \sec x}$$

2- **Exponential function** If is u any differentiable function of x then:

1) $\frac{d}{dx} = a^u = a^u \ln a \frac{du}{dx}$

2) $\frac{d}{dx} = e^u = e^u \frac{du}{dx}$

EXAMPLE 7: Find $\frac{dy}{dx}$ for the following function:

1. $y = 2^{3x}$

   $y' = 2^{3x} 3 \ln 2$

2. $y = (2^x)^2 \quad \Rightarrow \quad y = 2^{2x}$

   $y' = 2^{2x} \ln 2 (2) = 2^{2x+1} \ln 2$

Inverse function

1. **Trigonometric function**

(1) $\frac{d}{x}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

(2) $\frac{d}{x}\cos^{-1} u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

(3) $\frac{d}{x}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$

(4) $\frac{d}{x}\cot^{-1} u = -\frac{1}{1+u^2}\frac{du}{dx}$

(5) $\frac{d}{x}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad u > 1$

(6) $\frac{d}{x}\csc^{-1}u = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad u > 1$

EXAMPLE 1: Prove that $\frac{d}{x}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

Proof:

Let $y = \sin^{-1}x$

$$x = \sin y$$

$$\frac{d}{dx}x = \frac{d}{x}(\sin y)$$

$$1 = \cos y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\sin^2 y + \cos^2 y = 1$$

$$\cos^2 y = 1 - \sin^2 y$$

$$\cos y = \sqrt{1-\sin^2 y}$$

$$\cos y = \sqrt{1-x^2}$$

Hyperbolic function

If u is any differentiable function of x

1. $\frac{d}{x}\sinh u = \cosh u\frac{du}{dx}$

2. $\frac{d}{x}\cosh u = \sinh u\frac{du}{dx}$

3. $\frac{d}{x}\tanh u = \mathrm{sech}^2 u\frac{du}{dx}$

4. $\frac{d}{x}\coth u = -\operatorname{csch}^2 \frac{du}{dx}$

5. $\frac{d}{x}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$

6. $\frac{d}{x}\operatorname{csch} u = -\operatorname{csch} u \coth u \frac{du}{dx}$

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function:

1. $y = \coth(\tan x)$

Sol:

$$y' = -\operatorname{csch}^2(\tan x)\sec^2 x$$

2. $y = \sin^{-1}(\tanh x)$

Sol:

$$y' = \frac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\operatorname{sech}^2 x}{\sqrt{\operatorname{sech}^2 x}} = \operatorname{sech} x$$

<u>The Inverse hyperbolic function</u> If is u any differentiable function of x then:

1. $\frac{d}{x}\sinh^{-1} u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$

2. $\frac{d}{x}\cosh^{-1} u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$

3. $\frac{d}{x}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx} \qquad u < 1$

4. $\frac{d}{x}\coth^{-1} u = \frac{1}{1-u^2}\frac{du}{dx} \qquad u > 1$

5. $\frac{d}{x}\operatorname{sech}^{-1} u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$

6. $\frac{d}{x}\operatorname{csch}^{-1}u = -\frac{-1}{u\sqrt{1+u^2}}\frac{du}{dx}$

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $y = \cosh^{-1}(\sec x)$

Sol:

$$y' = \frac{\sec x \tan x}{\sqrt{(\sec x)^2 - 1}} = \frac{\sec x \tan x}{\sqrt{\tan^2 x}}$$

$$y' == \frac{\sec x \tan x}{\tan x} = \sec x \qquad \text{where } \tan x > 0$$

Application of derivative

1. **The slop of curve**

Secant to the curve is a line through two points on a curve.

Slope and tangent lines

1. We start with what we can calculate, namely the slope of the secant through P and a point Q nearby on the curve.
2. We find the limiting value of the secant slope (if it exists) as Q approaches P along the curve.
3. We take this number to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.

$$\text{The slop } m = f'(x) = \frac{dy}{dx}$$

EXAMPLE 1: Write an equation for the tangent line at $x = -1$ of the curve $f(x) = y = 4 - x^2$

Sol:

$$\frac{dy}{dx} = -2x$$

The slope at $x = -1$

$$y'\big|_{x=-1} = (-2x)_{-1}$$

$$= -2 \cdot (-1) = 2$$

$$m = 2$$

$$y = 4 - (-1)^2 = 3$$

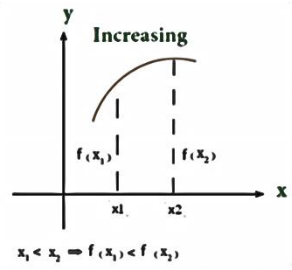
The line through $(-1,3)$ with slope $m = 2$

$$y - 3 = (2)(x - (-1))$$
$$y = 2x + 2 + 3$$
$$y = -x + 5$$

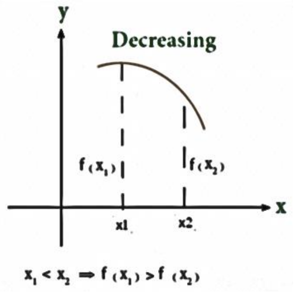
Increasing and decreasing function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ then f is said to be increasing on I.



2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$ then f is said to be decreasing on I.



First Derivative Test

1. If $f'(x) > 0$ at each point $\mathrm{x} \in (\mathrm{a, b})$, then f is increasing on $[\mathrm{a, b}]$.
2. If $f'(x) < 0$ at each point $\mathrm{x} \in (\mathrm{a, b})$, then f is decreasing on $[\mathrm{a, b}]$.

Definition Concave Up, Concave Down

The graph of a differentiable function $y = f(x)$ is

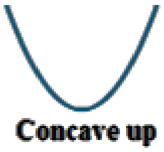
(a) Concave up on an open interval I if f' is increasing on I

(b) Concave down on an open interval I if f' is decreasing on I.

Second Derivative Test

1- If $f''(x) > 0$ on I, the graph of f over I Concave up

2- If $f''(x) < 0$ on I, the graph of f over I Concave down

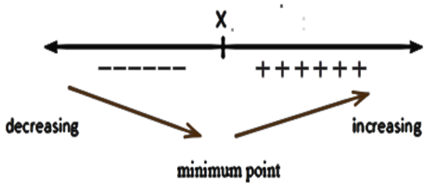


To find critical point (local maximum point and local minimum), concavity (Concave up and Concave down) and point of inflection point.

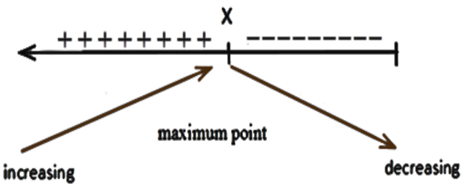
لايجاد النقاط الحرجة النهاية العظمى والصغرى والتحدب بنوعيه ونقطة الانقلاب.

1. First derivative test for local extrema f' or y'
2. $y' = 0$ the First derivative is zero at $x = ?$, find the value of x

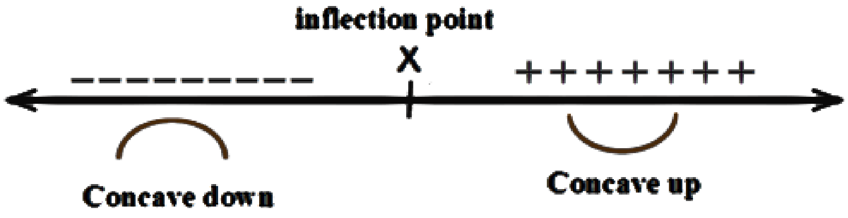
either



or



3. Second derivative test for concavity
4. Also $y'' = 0$ the second derivatives is zero that mean find value of x



EXAMPLE 1: Find all critical points, local minimum and maximum, concavity and inflection point. $y = 2x^3 - 3x^2 - 12x + 3$

Sol:

Test 1

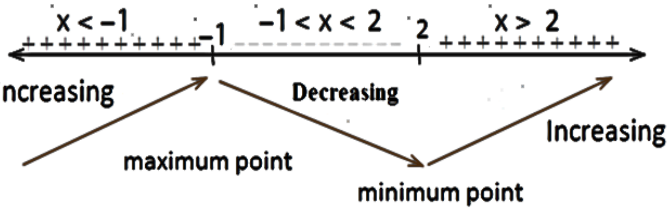
$y' = 0 \quad \Rightarrow \quad y' = 6x^2 - 6x - 12 = 0$

$\div 6 \;\; \Rightarrow \;\; x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x - 2 = 0 \;\; \Rightarrow x = 2$

$x + 1 = 0 \Rightarrow x = -1$



لكي نجد مناطف التزايد والتناقص نختبر المشتقة الاولى في خط الاعداد باخذ عدد اكبر من 2 ونلاحظ اشارة المشتقة وكذلك عدد في الفترة $(-1,2)$ وكذلك عدد عدد اقل من -1 كما في الرسم اعلاه

1. $\{x : x \in R \;\; , x > 2\}$ $\{x : x \in R \;\; , x < -1\}$ الدالة تكون متزايدة

2. $(-1,2)$ ومتناقصة في الفترة

Sub -1 in $y = 2x^3 - 3x^2 - 12x + 3$

$y = 2(-1)^3 - 3(-1)^2 - 12(-1) + 3 = 10$

$\mathrm{P} = (-1,10)$ **Maximum point** **نفطة عظمى**

Sub 2 in $y = 2x^3 - 3x^2 - 12x + 3$

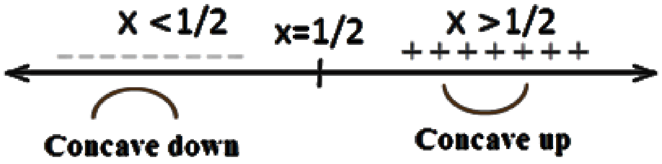
$y = 2(2)^3 - 3(2)^2 - 12(2) + 3 = -17$

$\mathrm{P} = (2,-17)$ **Minimum point** **نقطة صغرى**

Test 2

$y'' = 12x - 6 \qquad y'' = 0$

$12x - 6 = 0 \;\; \Rightarrow x = 1/2$



لكي نجد مناطف التفعر والتحدب نختبر المشتقة الثانية في خط الاعداد باخذ عدد اكبر من 1/ 2 ونلاحظ اشارة المشتقة وكذلك عدد اقل من 1/2 ونلاحض اشارة المشتقة كما في الرسم اعلاه

1. $\{x : x \in R \ \ , x > 1/2\}$ **Concave up** منطقة التفعر

2. $\{x : x \in R \ \ , x < 1/2\}$ **Concave down** منطقة النحدب

Sub 1/2 in $y = 2x^3 - 3x^2 - 12x + 3$

$y = 2(1/2)^3 - 3(1/2)^2 - 12(1/2) + 3 = -3.5$ at $x = 1/2$ $y = -3.5$

$\text{P} = (1/2, -3.5)$ **Inflection point** **نقطة الانقلاب**

Chapter five

Integration

Integration is the reversal of differentiation hence functions can be integrated by indentifying the anti-derivative.

Terminology

Indefinite and **Definite** integrals

There are two types of integrals: Indefinite and Definite.

Indefinite integrals are those with no limits and definite integrals have limits.

When dealing with indefinite integrals you need to add a constant of integration.

For example, if integrating the function f(x) with respect to x:

$$\int f(x)\ dx = g(x) + C$$

where $g(x)$ is the integrated function.

C is an arbitrary constant called the constant of integration.

dx indicates the variable with respect to which we are integrating, in this case, *x*.

The function being integrated, f(*x*), is called the **integrand**.

The Rule

1) **Constant Rule** $\boxed{\int a\, dx = ax + c}$ where a is constant

EXAMPLE: **1.** $\int 3\, dx = 3x + c$

2. $\int 4\, dy = 4y + c$

3. $\int \frac{7}{2} dz = \frac{7}{2} z + c$

2) Sum Rule $\boxed{\int (f \pm g)\, dx = \int f \;\; dx \pm \int g\, dx}$

EXAMPLE: **1.** $\int 3dx + 4dy = \int 3dx + \int 4dy = 3x + 4y + c$

3) The Power Rule $n \neq -1$ $\boxed{\int ax^n du = a\frac{x^{n+1}}{n+1} + c}$

EXAMPLE: **1.** $\int 4x^5 dx = 4\frac{x^6}{6} + c = \frac{2}{3}x^6 + c$

2. $\int 10x^{-5} dx = 10\frac{x^{-4}}{-4} + c = -\frac{5}{2}x^{-4} + c$

4) The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval *I* and f is continuous on *I*, then

$$\boxed{\int f(g(x))\, g'(x)\, dx = \int f(u)\, du}$$

EXAMPLES: **1.** $\int (x+1)^3 dx = \frac{(x+1)^4}{4} + c$

<u>Root function integral</u>

EXAMPLES: **1.** $\int 2x\sqrt{x^2 - 3}\, dx = \int (x^2 - 3)^{1/2}\, 2x\, dx$

$$\frac{(x^2-3)^{3/2}}{3/2} + c$$

$$\frac{2}{3}(x^2-3)^{3/2} + c$$

H.W **Evaluate**

1. $= \int \sqrt{x^2 - x^4}\, dx$

2. $\int (x^2 + 1)^2 (x+2)\, dx$

3. $\int \frac{2x-4}{\sqrt{x^2-4x+1}}dx$

تكامل الدوال المثلثية

Trigonometric function integral $\sin u$, $\cos u$

7. $\sin u$ $\qquad \frac{d}{x}\sin u = \cos u \frac{du}{dx}$

$$\boxed{\int \cos u\, du = \sin u + c}$$

8. $\cos u$ $\qquad \frac{d}{x}\cos u = -\sin u \frac{du}{dx}$

$$\boxed{\int \sin u\, du = -\cos u + c}$$

EXAMPLES: **1.** $\int \cos 3x\ dx = \frac{\sin 3x}{3} + c$

اذا كانت الدالة اسية ومشتقة داخل القوس متوفرة عندها نستخدم القوانين التالية

1. $\boxed{\int \sin^n au \cos au\, du = \frac{\sin^{n+1} au}{(n+1)a} + c}$

2. $\boxed{\int \cos^n au \sin au\, du = \frac{-\cos^{n+1} au}{(n+1)a} + c}$

EXAMPLE: **1.** $\int \sin^7 3x \cos 3x\ dx = \frac{\sin^8 3x}{(8)(3)} + c$

اذا كانت الدالة اسية والمشتقة غير متوفرة نتبع مايلي

1. اذا كانت الدالة اسية والمشتقة غير متوفرة وكان الاس عدد زوجي

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

EXAMPLES: **1.** $\int \sin^2 x\, dx = \int (\frac{1}{2} - \frac{1}{2}\cos 2x)\, dx$

$$= \int \frac{1}{2} dx - \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + c$$

2. اذا كانت الاس عدد فردي نستخدم القانون

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\cos^2 x = 1 - \sin^2 x$$

EXAMPLE 1 $\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx$

$$= \int (1 - \cos^2 x) \sin x\, dx$$

$$= \int (\sin x - \cos^2 x \sin x)\, dx$$

$$= \int \sin x\, dx - \int \cos^2 x \sin x\, dx$$

$$= -\cos x + \frac{\cos^3 x}{3} + c$$

اذا كان السؤال حاصل ضرب دالتين عندها نتبع القوانين التالية

1. اولا الاسس فردية عندها نفك الاس الفردي الاقل مرتبة ونحل حسب القوانين الفردية

EXAMPLE : $\int \sin^3 x \cos^5 x\, dx = \int \sin^2 x \sin x \cos^5 x\, dx$

$$= \int (1 - \cos^2 x) \sin x \cos^5 x\, dx$$

$$= \int (\sin x \cos^5 x - \cos^7 x \, \sin x) \, dx$$

$$= \int \cos^5 \, \sin x - \int \cos^7 x \, \sin x \, dx$$

$$= -\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + c$$

2. اذا كان احد الاس فردي والاخر زوجي عندها نفك الاس الفردي حسب القوانين الفردية

EXAMPLE : $\int \sin^5 x \cos^2 x \, dx = \int (\sin^2 x)^2 \, \sin x \cos^2 x \, dx$

$$= \int (1 - \cos^2 x)^2 \, \sin x \cos^2 x \, dx$$

$$= \int (1 - 2\cos^2 x + \cos^4 x) \, \sin x \, \cos^2 x \, dx$$

$$= \int \sin x \cos^2 x \, dx - \int 2\cos^4 x \sin x \, dx + \int \cos^6 x \sin x \, dx$$

$$= -\frac{\cos^3 x}{3} + 2\frac{\cos^5 x}{5} - \frac{\cos^7 x}{7} + c$$

اذا كان السؤال حاصل ضرب دالتين والزوايا مختلفة عندها نستخدم القوانين التالية

$\int \sin mx \sin nx \, dx$ $\int \sin mx \cos nx \, dx$ $\int \cos mx \cos nx \, dx$

1. $\sin mx \sin nx = \frac{1}{2}\cos(m-n)x - \frac{1}{2}\cos(m+n)x$
2. $\sin mx \cos nx = \frac{1}{2}\sin(m-n)x + \frac{1}{2}\sin(m+n)x$
3. $\cos mx \cos nx = \frac{1}{2}\cos(m-n)x + \frac{1}{2}\cos(m+n)x$

Evaluate

1. $\int \sin 7x \cos x \, dx$

$$= \int \frac{1}{2}(\sin 6x + \sin 8x) \, dx$$

$$= -\frac{1}{2}\frac{\cos 6x}{6} - \frac{1}{2}\frac{\cos 8x}{8} + c$$

$$= -\frac{1}{12}\cos 6x - \frac{1}{16}\cos 8x + c$$

ثانيا تكاملات

Integral $\tan x$, $\cot x$, $\sec x$ and $\csc x$

- **اولا تكاملات مباشرة**

1. If $y=\tan u \rightarrow y'=\sec^2 u\frac{du}{dx}$

$$\boxed{\int \sec^2 u\,du=\tan u+c}$$

2. If $y=\cot u \rightarrow y'=-\csc^2 u\frac{du}{dx}$

$$\boxed{\int \csc^2 u\,du=-\cot u+c}$$

3. If $y=\sec u \rightarrow y'=\sec u\tan u\frac{du}{dx}$

$$\boxed{\int \sec u\tan u\;du=\sec u+c}$$

4. If $y=\csc u \rightarrow y'=-\csc u\;\cot u\frac{du}{dx}$

$$\boxed{\int \csc u\cot u\;du=-\csc u+c}$$

- **ثانيا اذا كانت الدالة اسية المشتقة متوفرة عندها نستخدم القونين التالية**

$$\boxed{\int \tan^n au\sec^2 au\,du=\frac{\tan^{n+1}au}{(n+1)a}+c}$$

$$\boxed{\int \cot^n au\csc^2 au\,du=-\frac{\cot^{n+1}au}{(n+1)a}+c}$$

$$\boxed{\int \sec^n au\sec au\tan au\,du=\frac{\sec^{n+1}au}{(n+1)a}+c}$$

$$\boxed{\int \csc^n au\csc au\cot au\,du=-\frac{\csc^{n+1}au}{(n+1)a}+c}$$

1) $\int \tan x \sec^2 x\, dx = \frac{\tan^2 x}{2} + c$

- ثالثا اذا كانت الدالة اسية المشتقة غيرمتوفرة عندها نستخدم القونين التالية

$$\sec^2 x - \tan^2 x = 1$$
$$\tan^2 x = \sec^2 x - 1$$
$$\sec^2 x = 1 + \tan^2 x$$
$$\csc^2 x - \cot^2 x = 1$$
$$\csc^2 x = 1 + \cot^2 x$$
$$\cot^2 x = \csc^2 x - 1$$

EXAMPLE 1: $\int \tan^2 x\, dx = \int (\sec^2 x - 1) dx$

$$= \int \sec^2 x\, dx - \int dx$$

$$= \tan x - x + c$$

EXAMPLE 2: $\int \sec^4 x\, dx = \int \sec^2 x \sec^2 x\, dx$

$$= \int (1 + \tan^2 x) \sec^2 x\, dx$$

$$= \int \sec^2 x\, dx + \int \tan^2 x \sec^2 x\, dx$$

$$= \tan x + \frac{\tan^3 x}{3} + c$$

Transcendental Function Integral

Logarithm **exponential** **invers**

(4) If $y = \ln u$ $\Rightarrow y' = \frac{du}{u}$

$$y = \ln x \qquad \Rightarrow \frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}$$

$$\boxed{\int \frac{du}{u} = \ln|u| + c}$$

بمعنى ان مشتقة البسط = المقام فان التكامل هو المقام **ln**

EXAMPLE 1: $\int \frac{dx}{x} = \ln|x| + c$

2. $\int \frac{2x\,dx}{x^2+1} = \ln|x^2+1| + c$

Theorem

1. $\int \tan x = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + c$

2. $\int \cot x = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + c$

3. $\int \sec x = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= \ln|\sec x + \tan x| + c$$

Exponential function

1. If $y = e^u \qquad \to y' = e^u \frac{du}{dx}$

$$\boxed{\int e^u\,du = e^u + c}$$

2. $y = a^u \qquad \to y' = a^u \ln a \frac{du}{dx}$

$$\boxed{\int a^u\,du = \frac{a^u}{\ln a} + c}$$

EXAMPLE 1: $\int \frac{e^x}{1+3e^x}\,dx = \frac{1}{3}\int 3e^x(1+3e^x)^{-1}\,dx$

$$= \frac{1}{3}\int \frac{3e^x}{(1+3e^x)}dx$$

$$= \frac{1}{3}\ln\left|(1+3e^x)\right| + c$$

Invers function

1. If $y = \sin^{-1}u \quad \rightarrow \quad y' = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

$$\boxed{\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}x + c}$$

$$\boxed{\int \frac{du}{\sqrt{a^2-u^2}} = \begin{cases} \sin^{-1}\frac{x}{a} + c \\ -\cos^{-1}\frac{x}{a} + c \end{cases}}$$

2. $y = \tan^{-1}u \quad \rightarrow \quad y' = \frac{1}{1+u^2}\frac{du}{dx}$

$$\boxed{\int \frac{1}{1+u^2}du = \tan^{-1}x + c}$$

$$\boxed{\int \frac{1}{a^2+u^2}du = \begin{cases} \frac{1}{a}\tan^{-1}\frac{u}{a} \\ -\frac{1}{a}\cot^{-1}\frac{u}{a} \end{cases}}$$

Hyperbolic Functions

(1) If $y = \sinh u \qquad y' = \cosh u\frac{du}{dx}$

$y = \cosh u \qquad y' = \sinh u\frac{du}{dx}$

$y = \tanh u \qquad y' = \text{sech}^2 u\frac{du}{dx}$

1. $\int \sinh x\,dx = \cosh x + c$

2. $\int \cosh x\,dx = \sinh u + c$

3. $\int \text{sech}^2 x\,dx = \tanh x + c$

اذا كانت مشتقة الاس متوفرة نستخدم القوانين التالية

(2) $\int \sinh^{n} ax \cosh ax \, dx = \frac{\sinh^{n+1}}{a(n+1)} + c$

(3) $\int \cosh^{n} ax \sinh ax \, dx = \frac{\cosh^{n+1}}{a(n+1)} + c$

(4) If $\int \sinh^{n} x \, dx$ or $\int \cosh^{n} x \, dx$

اذا كانت الدالة اسية والمشتقة غير متوفرة وكان الاس عدد زوجي

Case 1: if n is even, we use identity $\cosh^2 x = \frac{\cosh 2x + 1}{2}, \ \sinh^2 x = \frac{\cosh 2x - 1}{2}$

اذا كانت الدالة اسية والمشتقة غير متوفرة وكان الاس عدد فردي

Case 2: if n is odd, we use identity $\cosh^2 x = \sinh^2 x + 1, \ \sinh^2 x = \cosh^2 x - 1$

<u>معكوس الدوال الزائدية</u>

(5) If $y = \sinh^{-1} u \quad \to \quad y' = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$

$$\boxed{\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1} \tfrac{u}{a} + c}$$

$$\boxed{\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1} \tfrac{u}{a} + c}$$

(6) If $y = \tanh^{-1} u \quad \to y' = \frac{1}{1-u^2} \frac{du}{dx}$

$$\boxed{\int \frac{1}{a^2 - u^2} du = \frac{1}{a} \tanh^{-1} u + c}$$

EXAMPLES: $\int \sinh^3 x \, dx$

$$= \int \sinh^2 x \sinh x \, dx$$

$$= \int (\cosh^2 x - 1) \sinh x \, dx$$

$$= \int \cosh^2 x \sinh x \, dx - \int \sinh x \, dx$$

$$= \frac{\cosh^3 x}{3} - \cosh x + c$$

H.W **Ex 1:** $\int \cosh^4 2x\,dx$ **Ex 2:** $\int \frac{x\,dx}{1-x^4}$

Ex 3: $\int \frac{xdx}{\sqrt{x^4-1}}$ **Ex 4:** $\int \frac{x^2dx}{\sqrt{1+x^6}}$

EXAMPLE: $\int e^x \sinh 2x\,dx$

Sol:

$$= \int e^x \left(\frac{e^{2x}-e^{-2x}}{2}\right)dx \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \int \frac{e^{3x}-e^{-x}}{2}\,dx$$

$$= \frac{1}{2}\int e^{3x}\,dx - \frac{1}{2}\int e^{-x}\,dx$$

$$= \frac{1}{2}\int \frac{1}{3}e^{3x}\,3dx + \frac{1}{2}\int e^{-x}(-dx)$$

$$= \frac{1}{2}\left[\frac{1}{3}e^{3x} + e^{-x}\right] + c$$

تكامل الدوال الزائدية

1. $\int \sinh x\,dx = \cosh x + c$

2. $\int \cosh x\,dx = \sinh x + c$

3. $\int \mathrm{sech}^2 x\,dx = \tanh x + c$

4. $\int \mathrm{csch}^2 x\,dx = -\coth x + c$

5. $\int \mathrm{sechx} \tanh x\,dx = -\mathrm{sech} x + c$

6. $\int \mathrm{cschx} \coth x\,dx = -\mathrm{csch} x + c$

تكامل معكوس الدوال الزائدية

1. $\int \frac{1}{\sqrt{1+x^2}}dx = \sinh^{-1} x + c$

2. $\int \frac{1}{\sqrt{x^2-1}}dx = \cosh^{-1} x + c$

3. $\int \frac{1}{1-x^2}\,dx = \begin{cases} \tanh^{-1} + c & \text{if} \quad |x| < 1 \\ \coth^{-1} + c & \text{if} \quad |x| > 1 \end{cases}$

$$= \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|$$

4. $\displaystyle \int \frac{1}{x\sqrt{1-x^2}}\ dx = -\operatorname{sech}^{-1} x + c$

5. $\displaystyle \int \frac{1}{x\sqrt{1+x^2}}\ dx = -\operatorname{csch}^{-1} x + c$

Methods of integration

طرق التكامل

1. Integration by parts — التكامل بالتجزئة

تبنى هذه الطريقة على قاعدة مشتقة حاصل ضرب دالتين

$$d(uv) = udv + vdu$$

$$udv = d(uv) - vdu$$

$$\boxed{\int udv = uv - \int vdu}$$

نلجا الى هذه الطريقة اذا لم نتمكن من الحل بالطرق السابقة وتمكنا من تجزئة السؤال الى جزئين احدهما قابل للتكامل والاخر قابل للاشتقاق. حيث يجب ان نحصل على $\int udv = uv - \int vdu$ الذي يفترض ان يكون ابسط من صيغة التكامل الاول في السؤال.

كيفية يتم اختيار $dv \, , \, u$

1. **الحالة الاولى** : اذا كان السؤال يحتوي على

1. Ln
2. invers

والمشتقة غير متوفرة عندها نختار u $\begin{matrix}\ln \\ \text{invers}\end{matrix}$ ثم نشتقها والباقي يكامل

Find the integration

Evaluate

EXAMPLE 1: $\displaystyle \int x \ln x \, dx$

$$u = \ln x \qquad du = \frac{1}{x} dx \qquad dv = x\,dx \qquad v = \int x\,dx = \frac{x^2}{2}$$

$$\int udv = uv - \int vdu$$

$$= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x\,dx \qquad = \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + c$$

EXAMPLE 2: $\int \ln x \, dx$

$$u=\ln x \qquad du=\frac{dx}{x} \qquad dv=dx \qquad v=x$$

$$\int u dv = u v - \int v du$$

$$= x\ln x - \int x \,\frac{dx}{x} \;= x\ln x - x + c$$

2. اذا لم تحتوي الدالة على ln او دالة معكوسة نختارالدالة التي اذا تم اشتقاقها لعدة مرات الى ان تصل الى الصفر هي u اما الباقي فهي dv

EXAMPLE 1: $\int x e^{x^2} dx = \frac{1}{2} e^{x^2} + c$

EXAMPLE 1: $\int x e^{x} dx$

$$u=x \qquad du=dx \qquad dv=e^x\, dx \qquad v=e^x$$

$$\int u dv = u v - \int v du$$

$$= x e^x - \int e^x dx = x e^x - e^x + c$$

H.W $\int x \sin^2 x \, dx$

3.طريقة الجدول: اذا لم تحتوي على ln و invers وهنالك دالة تحتاج الى عدد من الاشتقاقات لكي تصل الى الصفر

EXAMPLE 1: $\int x \cos x \, dx$

$$\int x\cos x\, dx = x\sin x + \cos x + c$$

التكامل المتكرر	الاشتقاق المتكرر
$\cos x$	x
$\sin x$	1
$-\cos x$	0

EXAMPLE 2: $\int x^3 e^x \, dx$

$$= x^3 e^x - 3x^2 e^x + 6x\, e^x - 6e^x + c$$

dv	u
e^x	x^3
e^x	$3x^2$
e^x	$6x$
e^x	6
e^x	0

(5) Trigonometric substitution integration التعويض باستخدام الدوال المثلثية

If we have

a^2-u^2	Special case	$a=1$	$1-u^2$
a^2+u^2	Special case	$a=1$	$1+u^2$
u^2-a^2	Special case	$a=1$	u^2-1

Case 1 في **الحالة الاولى** شكل a^2-u^2 ياخذ الصور التالية

a. a^2-u^2

b. $\sqrt{a^2-u^2}$

c. $(a^2-u^2)^n$

Let $u=a\sin\theta$ $\quad \frac{u}{a}=\sin\theta$ $\quad \theta=\sin^{-1}\frac{u}{a}$ $\quad du=a\cos\theta\,d\theta$

And used $\quad 1-\sin^2\theta=\cos^2\theta$

EXAMPLE: $\int\frac{x}{1-x^2}dx$ $\quad a^2-u^2 \quad a=1 \quad \theta=\sin^{-1}x$

$1-x^2 \quad u^2=x^2$

Let $x=a\sin\theta$ $\quad dx=\cos\theta\;d\theta$

$x=\sin\theta$

$$=\int\frac{\sin\theta\cos\theta}{1-\sin^2\theta}d\theta$$

$$=\int\frac{\sin\theta\cos\theta}{\cos^2\theta}d\theta$$

$$=\int\frac{\sin\theta}{\cos\theta}d\theta$$

$$=-\ln\left|\cos\theta\right|+c$$

$$=-\ln\left|\cos(\sin^{-1}x)\right|+c$$

EXAMPLE: $\int\frac{x}{\sqrt{1-x^2}}\;dx=-\frac{1}{2}\int-2x\;(1-x^2)^{-1/2}\;dx$

$$=-\frac{1}{2}\;\frac{(1-x^2)^{1/2}}{1/2}+c$$

or

Let $x=a\sin\theta$ $\quad dx=\cos\theta\;d\theta$ $\quad \theta=\sin^{-1}x$

$x=\sin\theta$

$$= \int \frac{\sin\theta\cos\theta}{\sqrt{1-\sin^2\theta}} d\theta$$

$$= \int \frac{\sin\theta\cos\theta}{\cos\theta} d\theta$$

$$= \int \sin\theta \, d\theta = \cos\theta + c$$

$$= \cos(\sin^{-1} x) + c$$

Prove $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$

Let $x = a\sin\theta \Rightarrow \frac{x}{a} = \sin\theta$ $\quad dx = a\cos\theta \, d\theta$ $\quad \theta = \sin^{-1}\frac{x}{a}$

$$= \int \frac{a\cos\theta}{\sqrt{a^2 - a^2\sin^2\theta}} d\theta$$

$$= \int \frac{a\cos\theta}{\sqrt{a^2(1-\sin^2\theta)}} d\theta$$

$$= \int \frac{a\cos\theta}{\sqrt{a^2\cos^2\theta)}} d\theta = \int d\theta = \theta + c$$

$$= \sin^{-1}\frac{x}{a} + c$$

Case (2) **الحالة الثانية**

a. $u^2 - a^2$

b. $\sqrt{u^2 - a^2}$

c. $(u^2 - a^2)^n$

Let $u = a\sec\theta$ $\quad \theta = \sec^{-1}\frac{u}{a}$ $\quad du = a\sec\theta\tan\theta \, d\theta$

And used $\quad \sec^2\theta - 1 = \tan^2\theta$

EXAMPLE 1: $\int \frac{dx}{\sqrt{x^2 - 1}} = \cosh^{-1} x + c$

Let $x = a\sec\theta$ $\quad a = 1$

$x = \sec\theta$ $\quad \theta = \sec^{-1} x$ $\quad dx = \sec\theta\tan\theta \, d\theta$

$$\int \frac{\sec\theta\tan\theta \, d\theta}{\sqrt{\sec^2\theta - 1}} = \int \sec\theta \, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + c$$

$$= \ln\left|\sec(\sec^{-1} x) + \tan(\sec^{-1} x)\right| + c$$

Case (3) **الحالة الثالثة**

a. $a^2 + u^2$

b. $\sqrt{a^2 + u^2}$

c. $(a^2 + u^2)^n$

Let $u = a\tan\theta$ $\qquad \theta = \tan^{-1}\frac{u}{a}$ $\qquad du = a\sec^2\theta\, d\theta$

And used $\qquad 1 + \tan^2\theta = \sec^2\theta$

EXAMPLE 1: $\displaystyle\int \frac{dx}{\sqrt{4 + x^2}} = \cosh^{-1}\frac{x}{2} + c$

Let $x = a\tan\theta$ $\qquad a = 2$

$x = 2\tan\theta$ $\qquad \theta = \tan^{-1}\frac{x}{2}$ $\qquad dx = 2\sec^2\theta\; d\theta$

$$\int \frac{2\sec^2\theta\, d\theta}{\sqrt{4 + 4\tan^2\theta}} = \int \frac{2\sec^2\theta\, d\theta}{2\sqrt{1 + \tan^2\theta}}$$

$$= \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + c$$

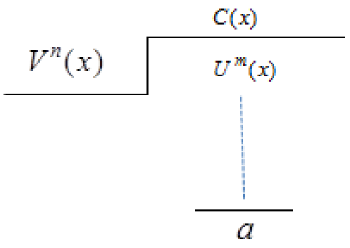
$$= \ln\left|\sec(\tan^{-1}\tfrac{x}{2}) + \tan(\tan^{-1}\tfrac{x}{2})\right| + c$$

3. Partial fractions integration

التكامل بالكسورالجزيئية

1. اولا

$\displaystyle\int \frac{U^m(x)}{V^n(x)}\, dx$ $\qquad$ if $m \geq n$



Used

$$\int \frac{U^m(x)}{V^n(x)}\, dx = \int \left(C(x) + \frac{a}{V(x)}\right) dx$$

EXAMPLE 1: $\int \frac{x\,dx}{1+x}$

$$\begin{array}{r|l} & \quad 1 \\ \hline 1+x & \quad x \\ & \mp x \mp 1 \\ \cline{2-2} & \quad -1 \end{array}$$

$$\int (1+\frac{-1}{1+x})\,dx$$

$$\int dx - \int \frac{dx}{1+x}$$

$$x-\ln|1+x|+c$$

ثانيا

$$\int \frac{U(x)}{V(x)}dx$$

If $V(x)=(x+a)(x+b(x+c)(x+E)$........

$$\frac{U(x)}{V(x)}=\frac{U(x)}{(x+a)(x+b)(x+c)(x+E)}$$

$$\frac{U(x)}{V(x)}=\frac{A}{(x+a)}+\frac{B}{(x+b)}+\frac{C}{(x+c)}+\frac{D}{(x+E)}$$

If $V(x)=(x^2+a)(x^2+b)(x^2+c)$.......

$$\frac{U(x)}{V(x)}=\frac{U(x)}{(x^2+a)(x^2+b)(x^2+c)}=$$

$$\frac{U(x)}{V(x)}=\frac{Ax+B}{(x^2+a)}+\frac{Cx+D}{(x^2+b)}+\frac{Ex+f}{(x^2+c)}$$

If $V(x)=(x+a)^n$

$$\frac{U(x)}{(x+a)^n}=\frac{A}{(x+a)}+\frac{B}{(x+a)}+\cdots\cdots\cdots\cdot+\frac{Z}{(x+a)^n}$$

And last step must be find the values of A,B,C------and etc.

Evaluate

1. $\int \frac{2x+9}{x^2-9}dx=\int \frac{2x+9}{(x-3)(x+3)}dx=\int (\frac{A}{(x-3)}+\frac{B}{(x+3)})\,dx$

$$\frac{2x+9}{x^2-9}=\frac{A}{(x-3)}+\frac{B}{(x+3)}$$

$$\frac{2x+9}{(x+3)(x-3)}=\frac{A(x+3)+B(x-3)}{(x-3)(x+3)}$$

$$2x+9=Ax+3A+Bx-3B$$
$$2x+9=Ax+Bx+3A-3B$$
$$2x=(A+B)x$$
$$9=3(A-B)$$
$$A+B=2$$
$$A-B=3$$
$$2A=5$$
$$A=\tfrac{5}{2}$$
$$\tfrac{5}{2}+B=2$$
$$B=2-\tfrac{5}{2}$$
$$B=-\tfrac{1}{2}$$
$$=\int(\frac{A}{(x-3)}+\frac{B}{(x+3)})\,dx$$
$$=\int(\frac{(5/2)}{(x-3)}+\frac{(-1/2)}{(x+3)})\,dx$$
$$=\frac{5}{2}\ln|(x-3)|-\frac{1}{2}\ln|(x+3)|+c$$

طريقة (z)

تستخدم هذه الطريقة في حالة احتواء السؤال دوال مثلثية فقط

$$\text{Let }\sin x=\frac{2z}{1+z^2}$$
$$(1-z^2)^2+(2z)^2=(1+z^2)^2$$
$$1-2z^2+z^4+4z^2=1+2z^2+z^4$$
$$1+4z^2-2z^2+z^4=1+2z^2+z^4$$
$$1+2z^2+z^4=1+2z^2+z^4$$

$$\sin x=\frac{2z}{1+z^2}\qquad \cos x=\frac{1-z^2}{1+z^2}\qquad \sec x=\frac{1}{\cos x}\qquad \sec x=\frac{1+z^2}{1-z^2}$$
$$\csc x=\frac{1+z^2}{2z}\qquad \tan x=\frac{2z}{1-z^2}\qquad \cot x=\frac{1-z^2}{2z}$$
$$z=\tan\frac{x}{2}$$
$$\tan^{-1}z=\frac{x}{2}$$

$$dx = \frac{2\,dz}{1+z^2}$$

EXAMPLE 1: $$\int \frac{dx}{1+\sin x} = \int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}}$$

$$= \int \frac{2z/1+z^2}{\frac{1+z^2+2z}{1+z^2}}\,dz$$

$$= \int \frac{2\,dz}{1+z^2+2z} = \int \frac{2\,dz}{z^2+2z+1}$$

$$= \int \frac{2\,dz}{(z+1)(z+1)} = \int \frac{2\,dz}{(z+1)^2}$$

$$= \int \frac{2\,dz}{(z+1)(z+1)} = \int 2(z+1)^{-2}\,dz$$

$$= 2\int (z+1)^{-2}\,dz$$

$$2\frac{(z+1)^{-1}}{-1}+c$$

$$2\frac{(\tan\frac{x}{2}+1)^{-1}}{-1}+c$$

طريقة الفرضية

اذا لم نستطع الحل بكل الطرق السابقة وكانت الدالة معقدة

EXAMPLE : $$\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})}$$

نفرض ان

$$y = \sqrt{x}$$

$$x = y^2$$

$$dx = 2y\,dy$$

$$= \int \frac{2y\,dy}{y(1+y)} = \int \frac{2dy}{1+y}$$

$$= 2\int \frac{dy}{1+y}$$

$$= 2\ln\left|1+y\right|+c$$

$$= 2\ln\left|1+\sqrt{x}\right|+c$$

طريقة اكمال المربع

اذا كان السؤال **لايمكن تحليله** ax^2+bx+c معامل $x^2=1$

$a(x^2+\frac{b}{a}x+\frac{c}{a})$ نضيف ونطرح نصف $\frac{x}{2}$ أي ان $(\frac{b}{2a})^2$

EXAMPLE 1: $\int \frac{dx}{x^2+2x+2}$

$$\int \frac{dx}{x^2+2x+1-1+2}=\int \frac{dx}{(x+1)^2+1}$$

$$=\tan^{-1}(x+1)+c$$

The Area (المساحة)

من التطبيقات المهمة للتكامل المحدد هو ايجاد المساحة تحت منحني الدالة $y=f(x)$ حيث ان $f(x)$ دالة مستمرة في الفترة $[a,b]$

1. المساحة المحددة بمنحني الدالة $y=f(x)$ ومحور السينات x-axis لايجاد ذلك نتبع الخطوات التالية.

نقاطع المنحني مع محور السينات x-axis وذلك بجعل $y=0$ لمعرفة $f(x)>0$ او $f(x)<0$

عندما $f(x)>0$ المساحة تساوي $A=\int_a^b f(x)\,dx$

وعندما $f(x)<0$ فان المساحة تستوي $A=-\int_a^b f(x)\,dx$

EXAMPLE: finds the area bounded by the x-axis and the curve $y=2x-x^2$

Sol:

$$y=2x-x^2$$
$$y=0$$
$$2x-x^2=0$$
$$x(2-x)=0$$

Either $x=0$ or $2-x=0$ $x=2$

$$\int_0^2 (2x-x^2)\,dx$$

$$\frac{2x^2}{2}\Big|_0^2-\frac{x^3}{3}\Big|_0^2$$

$$x^2\Big|_0^2-\frac{x^3}{3}\Big|_0^2=(4-0)-(\frac{8}{3}-0)$$

$$4-\frac{8}{3}=\frac{4}{3}$$

EXAMPLE: finds the area bounded by the y-axis and the curve $x=y^2-y^3$

Sol:

لاحظ ان في السؤال مطلوب المساحة تحت المنحي مع المحور y y-axis هنا نجعل $x=0$

$$x=y^2-y^3$$

$$y^2-y^3=0$$

$$y^2(1-y)=0$$

$$y^2=0 \qquad y=0 \qquad \text{or} \qquad 1-y=0 \qquad y=1$$

$$A=\int_0^1 x\,dy$$

$$A=\int_0^1 (y^2-y^3)\,dy$$

$$\left.\frac{y^3}{3}\right|_0^2-\left.\frac{y^4}{4}\right|_0^2$$

$$\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$$

2. المساحة المحصورة بين دالتين

نجد نقاط التقاطع لايجاد حدود التكامل

$$A=\int_a^b (y_2-y_x)\,dx \qquad \text{if} \quad y_2>y_1$$

$$A=\int_a^b (y_1-y_2)\,dx \qquad \text{if} \quad y_1>y_2$$

EXAMPLE: finds the area bounded by the curve $y=x^2$ and the line $y=x$

Sol:

$$x^2-x=0 \qquad x(x-1)=0 \ \text{Either} \qquad x=0 \qquad \text{or} \qquad x-1=0 \qquad x=1$$

لاحظ خلال الفترة $[0, 1]$ ان $x>x^2$

$$\int_a^b (y_2-y_1)\,dx=\int_0^1 (x-x^2)\,dx=\left.\frac{x^2}{2}-\frac{x^3}{3}\right|_0^1=[\frac{1}{2}-\frac{1}{3}]-[0-0]=\frac{1}{6}$$