Definition:-

Let X be a real or complex vector space over F where F is a field of real number R or complex number ₡ A mapping $< \, , >$: $X \times X \rightarrow F$ is called inner product on X . if it's satisfy the following properties :

1- $<x,y> = \overline{<y,x>}$ $\forall x,y \in X$.
2- $<x+y,z> = <x,z> + <y,z>$ $\forall x,y,z \in X$.
3- $<\lambda x,y> = \lambda <x,y>$ $\forall x,y \in X$, $\lambda \in F$.
4- $<x,x> > 0$ when $X \neq 0$.

Definition:-

An inner product space is a vector space X with inner product defined on X . Then (x, < , >) is inner product space.

Example:-

Let X= $₡^n$, The set of all n- tuples of complex number X= $(\alpha_1,\alpha_2 ,\ldots, \alpha_n)$,

y $=(\beta_1,\beta_2,\ldots,\beta_n)$ where α_ι, β_ι are complex number

define $<x,y> = \sum_{i=1}^{n} \alpha_\iota \overline{\beta_\iota}$

Then the order pair ($₡^n$, < , >) is an inner product space.

Solution:-

1- $\langle x,y \rangle = \sum_{i=1}^{n} \propto \iota \overline{\beta \iota}$

$\overline{\langle y,x \rangle} = \overline{\sum_{i=1}^{n} \beta\iota \, \overline{\alpha\iota}} = \sum_{i=1}^{n} \overline{\beta\iota} \, \overline{\overline{\alpha\iota}} = \sum_{i=1}^{n} \overline{\beta\iota} \, \alpha\iota = \sum_{i=1}^{n} \propto \iota \overline{\beta\iota}$

$\therefore \langle x,y \rangle = \overline{\langle y,x \rangle}$.

2- $\langle x+y , z \rangle = \sum_{i=1}^{n} (\alpha\iota + \beta\iota) \overline{\gamma\iota}$ Where $Z \in \mathbb{C}^n$, $z = (\gamma_1, \gamma_2, \ldots, \gamma_n)$

$= \sum_{i=1}^{n} (\alpha\iota \, \overline{\gamma\iota} + \beta\iota \, \overline{\gamma\iota}) = \sum_{i=1}^{n} \alpha\iota \, \overline{\gamma\iota} + \sum_{i=1}^{n} \beta\iota \, \overline{\gamma\iota}$

$= \langle x,z \rangle + \langle y,z \rangle$.

3- $\langle \lambda x,y \rangle = \sum_{i=1}^{n} \lambda \, \alpha\iota \overline{\beta\iota}$

$= \lambda \sum_{i=1}^{n} \alpha\iota \overline{\beta\iota}$

$= \lambda \langle x,y \rangle$

4- $\langle x,x \rangle = \sum_{i=1}^{n} \propto \iota \, \overline{\alpha\iota}$

$= \sum_{i=1}^{n} \left| \propto \iota \right|^2 > 0 \longrightarrow \langle x , x \rangle > 0$

When $x \neq \theta$

$\therefore (\mathbb{C}, \langle \; , \; \rangle)$ is inner product space.

Example:-

Let X= c[a,b] , The set of all continuous function defined on closed interval [a,b] with vector addition $(f+g)_{(x)} = f(x)+ g(x)$.

Scalar multiplication $(\alpha f)_{(x)} = \alpha f(x)$.

Define <f,g> = $\int_a^b f(x)\,.\,\overline{g(x)}$ dx then (x, < ,>) is an inner product space .

Solution:-

1- $<f,g> = \int_a^b f(x)\,.\,\overline{g(x)}\, dx$

$\overline{<g,f>} = \overline{\int_a^b g(x)}\;.\,\overline{f(x)}\; dx$

$=\int_a^b \overline{g(x)}\;\;.\,\overline{\overline{f(x)}}\; dx$

$= \int_a^b \overline{g(x)}.\, f(x)\; dx$

$= \int_a^b f(x)\,.\,\overline{g(x)}\; dx$

$\therefore <f,g> = \overline{<g,f>}$.

2- $< f+g ,h > = \int_a^b ((f+g)_{(x)})\,.\overline{h(x)}\;\; dx$

$= \int_a^b (f(x)+g(x))\,.\,\overline{h(x)}\;\; dx$

$= \int_a^b f(x)\,.\,\overline{h(x)}\;\; dx + \int_a^b g(x)\,.\overline{h(x)}\;\; dx$

$= <f,h> + <g,h>$.

3- $<\lambda f,g> = \int_a^b (\lambda f)_{(x)}\;.\,\overline{g(x)}\;\; dx = \lambda \int_a^b f(x)\,.\,\overline{g(x)}\; dx$

$= \lambda < f, g >$.

4- $< f,f >$ when $f \neq 0$

$< f, f> = \int_a^b f(x) \,.\, \overline{f(x)}\, dx$

$= \int_a^b |f(x)|^2 \; dx > 0$

$\therefore <f, f> \; > 0$

$\therefore$ (x , < , >) is inner product space.

Remark:-

Let X be inner product space then any subspace of X is also inner product space.

Theorem:-

In any inner product space then :

1- $< x,y+z > = <x,y> + <x,z>$.
2- $< x, \lambda y> = \bar{\lambda} < x,y>$.
3- $<\theta ,y> = < x,\theta> = 0$.
4- $<x-y,z> = < x,z> - < y,z >$.
5- $< x ,y-z > = < x,y> - < x, z >$.
6- $< x,z> = < y, z>$ for all z then x=y.

Proof :-

1- $< x,y+z > \overline{< y + z , x >} = \overline{< y , x > + < z, x >}$
$= \overline{<y, x>} + \overline{<z + x>} = <x,y>+<x,z>$.

2- $<x,\lambda y> = \overline{<\lambda y, x>} = \bar{\lambda}\, \overline{<y, x>} = \bar{\lambda} <x, y>$.

3- $<\theta, y> = <\theta + \theta,y> = <\theta,y>+<\theta,y>$
$\Rightarrow <\theta,y> - <\theta,y> = <\theta,y>$
$\Rightarrow 0 = <\theta,y>$

$\Rightarrow <\theta,y> = 0$

$< x,\theta> = < x,\theta+\theta> = <x,\theta> + <x,\theta>$

$\Rightarrow <x,\theta> - <x,\theta> = <x,\theta>$

$\Rightarrow 0 = <x,\theta>$

$\Rightarrow <x,\theta> = 0.$

4- $< x-y ,z> = < x+(-y) ,z> = <x,z> + <-y, z>$

$= <x,z> - <y ,z>$.

5- $< x,y-z> = <x ,y+(-z)> = <x,y> + <x,-z>$

$= <x,y> - < x,z>$.

6- $<x,z > = <y,z>$

$\Rightarrow <x,z> - <y,z> = 0$

$\Rightarrow < x-y,z> = 0$ for all z

$\Rightarrow x-y = 0$

$\Rightarrow x=y$.

Exercises:-

Let $X = R^2$, $x = (x_1,x_2)$, $y = (y_1,y_2)$

Show that whether the function are inner product on X or not

1- $<x,y> = x_1y_1 + x_2y_2$
2- $<x,y> = 3x_1y_1 + x_2y_2$
3- $<x,y> = x_1^2y_1^2 + x_2^2y_2^2$

Proposition:- In any inner product space

1- $< \sum_{k=1}^{n} \alpha_k \,.\, x_k , y > = \sum_{k=1}^{n} \alpha_k < x_k , y >$

2- $< x , \sum_{k=1}^{n} \alpha_k \; y_k > = \sum_{k=1}^{n} \overline{\alpha_k} < x, y_k >$

Proof:-

1- To prove that by using mathematical induction

a) To prove it's true when n=1

$< \sum_{k=1}^{1} \alpha_k \; x_k , y > = < \alpha_1 \; x_1 , y > = \alpha_1 < x_1, y >$

$\sum_{k=1}^{1} \alpha_k < x_k , y > = \alpha_1 < x_1, y >$

$\therefore$ it's true when n=1

b) Suppose its true when n=m

Then $< \sum_{k=1}^{m} \alpha_k \; x_k , y > = \sum_{k=1}^{m} \alpha_k < x_k , y >$

c) To prove its true when n=m+1

$< \sum_{k=1}^{m+1} \alpha_k \; x_k , y > = < \sum_{k=1}^{m} \alpha_k \; x_k + \alpha_{m+1} \; x_{m+1} , y >$

$= < \sum_{k=1}^{m} \alpha_k \; x_k , y > + < \alpha_{m+1} \; x_{m+1} , y >$

[since $< x+y , z> = < x,z> + < y,z>$]

$= \sum_{k=1}^{m} \alpha_k < x_k , y > + \alpha_{m+1} < x_{m+1} , y >$

[by (b) and $< \lambda x , y > = \lambda < x,y >$]

$= \alpha 1 < x_1 , y > + \alpha_2 < x_2 , y > + \ldots\ldots + \alpha_{m+1} < x_{m+1} , y >$

$= \sum_{k=1}^{m+1} \alpha_{m+1} < x_{m+1} , y >$

Definition:- **Let X be inner product space the norm of a vector x ∈X is defined by**

$\|x\|=\sqrt{< x,x >}$.

Example:-

In the unitary space $\mathbb{C}^n$ if x= λ_k , Then

$\|x\| = (\sum_{k=1}^{n} |\lambda_k|^2)^{1/2}$.

Solve:-

In the unitary $\mathbb{C}^n$ if x= λ_k and y= μ_k

$<x,y> = \sum_{k=1}^{n} \lambda_k \, \overline{\mu}_k$

$\|x\| = \sqrt{< x,x >} = (\sum_{k=1}^{n} \lambda_k \, \overline{\lambda}_k)^{1/2} = (\sum_{k=1}^{n} |\lambda_k|^2)^{1/2}$

Example:-

In the inner product space C[a,b] , defined by

$<f,g> = \int_a^b f(x)\overline{g(x)}$ dx then

$\|f\| = (\int_a^b |f(x)|^2 \, dx)^{1/2}$

Solve:-

$\|f\| = \sqrt{< f,f >} = (\int_a^b f(x) . \overline{f(x)} \, dx)^{1/2}$

$= (\int_a^b |f(x)|^2 \, dx)^{1/2}$

Theorem:-

In inner product space

1) $\|\lambda x\| = |\lambda| . \|x\|$

2) $\|x\| > 0$ when $x \neq \theta$; $\|x\| = 0$ if and only if $x=\theta$.

Proof:-

1) $\|\lambda x\|^2 = <\lambda x, \lambda x> = \lambda\bar{\lambda} <x,x> = |\lambda|^2 \|x\|^2$

$\therefore \|\lambda x\|^2 = |\lambda|^2 . \|x\|^2 \Rightarrow \|\lambda x\| = |\lambda| . \|x\|$.

2) When $x\neq\theta \Rightarrow <x,x> > 0 \Rightarrow \sqrt{<x,x>} > 0 \Rightarrow \|x\| > 0$

If $x=\theta \Rightarrow <x,x> = 0 \Rightarrow \sqrt{<x,x>} =0 \Rightarrow \|x\|=0$

$\Leftarrow$ if $\|x\| = 0 \Rightarrow \sqrt{<x,x>} = 0 \Rightarrow <x,x> =0 \Rightarrow x= \theta$.

Theorem:- ((parallelogram Law))

In inner product space then

$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

Proof :-

$\|x+y\|^2 = <x+y ,x+y>$

$= <x,x> + <x,y> + <y,x> + <y,y>$----------(1)

$\|x-y\|^2 = <x-y , x-y>$

$= <x,x> -<x,y> - <y,x> + <y,y>$ ---------(2)

$\therefore\ \|x+y\|^2 + \|x-y\|^2 = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle$

$= 2\langle x,x\rangle + 2\langle y,y\rangle$

$= 2\|x\|^2 + 2\|y\|^2$.

Theorem :- ((Polarization identity))

In inner product space then

$$\langle x,y\rangle = \tfrac{1}{4}\{ \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 \}$$

Proof:-

$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle$

$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - \langle x,y\rangle - \langle y,x\rangle$

$\|x+iy\|^2 = \|x\|^2 + \|y\|^2 - i\langle x,y\rangle + i\langle y,x\rangle$

$\|x-iy\|^2 = \|x\|^2 + \|y\|^2 + i\langle x,y\rangle - i\langle y,x\rangle$

$\tfrac{1}{4}[\|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle - \|x\|^2 - \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle + i\|x\|^2 + i\|y\|^2 + \langle x,y\rangle - \langle y,x\rangle - i\|x\|^2 - i\|y\|^2 + \langle x,y\rangle - \langle y,x\rangle]$

$\tfrac{1}{4}[4\langle x,y\rangle] = \langle x,y\rangle$.

Theorem:- (Cauchy – Shwarz inequality) :

In any inner product space $|\langle x,y\rangle| \le \|x\| . \|y\|$

Corollaries :-

1) If $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots\ldots, \beta_n$ are complex number then

$$|\sum_{i=1}^{n} \alpha_i \ \bar{\beta}_i| \leq (\sum_{i=1}^{n} |\alpha_i|^2)^{1/2} \ . (\sum_{i=1}^{n} |\beta_i|^2)^{1/2} .$$

2) If f and g are continues complex valued function on C [a,b] then

$$| \int_a^b f(x).\overline{g(x)} \ dx |^2 \leq \int_a^b |f(x)|^2 \, dx \ . \int_a^b |g(x)|^2 \, dx$$

Proof:-

1) $<x,y> = \sum_{i=1}^{n} \alpha_i \ \bar{\beta}_i \Rightarrow |<x,y>| = |\sum_{i=1}^{n} \alpha_i \ \bar{\beta}_i|$

$\|x\| = \sqrt{<x,x>} = \sqrt{\sum_{i=1}^{n} \alpha_i \ \bar{\alpha}_i}$

$= \sqrt{\sum_{i=1}^{n} |\alpha_i|^2} = (\sum_{i=1}^{n} |\alpha_i|^2)^{1/2}$

$\|y\| = \sqrt{<y,y>} = \sqrt{\sum_{i=1}^{n} \beta_i \ \bar{\beta}_i}$

$= \sqrt{\sum_{i=1}^{n} |\beta_i|^2} = (\sum_{i=1}^{n} |\beta_i|^2)^{1/2}$

By Cauchy – Shwarz inequality $|<x,y>| \leq \|x\| . \|y\|$

$\Rightarrow |\sum_{i=1}^{n} \alpha_i \ \bar{\beta}_i| \leq (\sum_{i=1}^{n} |\alpha_i|^2)^{1/2} \ . (\sum_{i=1}^{n} |\beta_i|^2)^{1/2}$

2) $<f,g> = \int_a^b f(x). \overline{g(x)} \ dx$

$\Rightarrow |<f,g>|^2 = |\int_a^b f(x). \overline{g(x)} \ dx|^2$

$\|f\| = \sqrt{< f,f >} = \sqrt{\int_a^b f(x).\overline{f(x)}\, dx} = \sqrt{\int_a^b |f(x)|^2\ dx}$

$\Rightarrow \|f\|^2 = \int_a^b |f(x)|^2$ dx

$\|g\| = \sqrt{< g,g >} = \sqrt{\int_a^b g(x).\overline{g(x)}\, dx} = \sqrt{\int_a^b |g(x)|^2\ dx}$

$\Rightarrow \|g\|^2 = \int_a^b |g(x)|^2$ dx

By Cauchy – Shwarz inequality $|<x,y>| \leq \|x\| .\|y\|$

$\Rightarrow |< x,y>|^2 \leq \|x\|^2 .\|y\|^2$

$\therefore\ |<f,g>|^2 \leq \|f\|^2 .\|g\|^2$

$\Rightarrow |\int_a^b f(x).\, \overline{g(x)}\ dx\ |^2$

$\leq \int_a^b |f(x)|^2\ dx\ .\int_a^b |g(x)|^2\ dx$

Theorem:- (Triangle- inequality) :-

In inner product space $\|x+y\| \leq \|x\| + \|y\|$

Proof:-

$\|x+y\| = \sqrt{< x + y, x + y >}$

$\|x+y\|^2 = <x+y, x+y>$

$= <x,x> + <x,y> + <y,x> + <y,y>$

$= <x,x> + <x,y> + \overline{< x,y >} + <y,y>$

[since $< x,y > = \overline{< y,x >}$]

$= <x,x> + 2Re <x,y> + <y,y>$

$\leq\ <x,x> + 2|<x,y>| + <y,y>$

$\leq \|x\|^2 + 2(\|x\| . \|y\|) + \|y\|^2$

[by Cauchy – Shwarz inequality $|<x,y>| \leq \|x\| . \|y\|$

and $<x,x> = \|x\|^2$, $\|y\|^2 = <y,y>$]

$\leq (\|x\| + \|y\|)^2$

$\therefore \|x+y\|^2 \leq (\|x\| + \|y\|)^2 \Rightarrow \|x+y\| \leq \|x\| + \|y\|$

Corollary:- if f and g are continuous complex valued function on [a,b] then:

$$\left(\int_a^b |f(x)+g(x)|^2\,dx\right)^{1/2} \leq \left(\int_a^b |f(x)|^2 dx\right)^{1/2} + \left(\int_a^b |g(x)|^2 dx\right)^{1/2}$$

$$\| f+g \| = \sqrt{< f+g\,, f+g >} = \sqrt{\int_a^b |f(x)+g(x)|^2\,dx}$$

$$= \left(\int_a^b |f(x)+g(x)|^2\,dx\right)^{1/2}$$

$$\|f\| = \sqrt{< f, f >} = \sqrt{\int_a^b |f(x)|^2\,dx} = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2}$$

$$\|g\| = \sqrt{< g, g >} = \sqrt{\int_a^b |g(x)|^2\,dx} = \left(\int_a^b |g(x)|^2\,dx\right)^{1/2}$$

By triangle inequality $\|x+y\| \leq \|x\| + \|y\|$

$\Rightarrow \|f+g\| \leq \|f\| + \|g\|$

$$\Rightarrow \left(\int_a^b |f(x)+g(x)|^2\,dx\right)^{1/2} \leq \left(\int_a^b |f(x)|^2 dx\right)^{1/2} + \left(\int_a^b |g(x)|^2 dx\right)^{1/2}$$

Definition:-

An inner product space which is complete to metric drivetive from inner product is called Hilbert space.

Remark:-

Complete inner product space is called Hilbert space.

Example:- Let $\mathbb{C}^n$ the space of all n-tuples of complex

Let x= $(\alpha_1,\alpha_2,\ldots,\alpha_n)$

y= $(\beta_1,\beta_2,\ldots,\beta_n)$

Define inner product by $<x,y> = \sum_{i=1}^{n} \alpha_i \bar{\beta_i}$ then the space $\mathbb{C}^n$ with inner product defined is Hilbert space.

Sol:-

1) To prove $<x,y>$ is inner product.
2) To prove $\mathbb{C}^n$ is complete .

1) $<x,y> = \sum_{i=1}^{n} \alpha_i \bar{\beta_i}$

$\overline{< y,x >} = \overline{\sum_{i=1}^{n} \beta_i \bar{\alpha_i}} = \sum_{i=1}^{n} \bar{\beta_i}\bar{\bar{\alpha_i}} = \sum_{i=1}^{n} \bar{\beta_i}\alpha_i = \sum_{i=1}^{n} \alpha_i\bar{\beta_i}$

$\therefore <x,y> = \overline{< y,x >}$.

2) $<x+y ,z> = \sum_{i=1}^{n}(\alpha_i + \beta_i)\bar{\gamma_i} = \sum_{i=1}^{n}(\alpha_i\bar{\gamma_i} + \beta_i\bar{\gamma_i})$

$= \sum_{i=1}^{n} \alpha_i \bar{\gamma_i} + \sum_{i=1}^{n} \beta_i \bar{\gamma_i}$

$<x,z> + <y,z>$.

3) $<\lambda x,y> = \sum_{i=1}^{n} \lambda\alpha_i \bar{\beta_i} = \lambda\sum_{i=1}^{n} \alpha_i \bar{\beta_i} = \lambda<x,y>$.

4) $<x,x> = \sum_{i=1}^{n} \alpha_i \bar{\alpha_i} = \sum_{i=1}^{n} |\alpha_i|^2 > 0$ when $x\neq \theta$

$\therefore <x,x> > 0$

$\therefore (\mathbb{C}^n , < , >)$ is inner product space.

Exercises:- To prove $\not{C}^n$ is complete.

Remark:-

Every Hilbert space is inner product space but the converse is not necessary to be true see the following example.

Example:-

Let x= c[a,b] for f, g $\in$ x

$\langle f,g\rangle = \int_a^b f(x)\,.\,\overline{g(x)}\,dx$

Then (x, < , >) is an inner product space which is not Hilbert space.

Sol:-

To prove $\langle f,g\rangle$ is inner product

Let $f_n:[0,2] \to |R$ defined by :

$\forall n \in N$

Let $f_n(x) = \begin{cases} x^n & x < 1 \\ 1 & x \geq 1 \end{cases} \qquad 0 \leq x \leq 2$

$f_0(x) = \begin{cases} 0 & x<1 \\ 1 & x \geq 1 \end{cases} \qquad 0 \leq x \leq 2$

if m > n $\Rightarrow \|f_n - f_m\| = \int_0^2 |f_n - f_m|$

$= \int_0^1 |f_n - f_m| + \int_1^2 |f_n - f_m|$

$= \int_0^1 |x^n - x^m| + \int_1^2 |1 - 1| = \frac{x^{n+1}}{n+1} - \frac{x^{m+1}}{m+1} \Big|_0^1 + 0$

$= \frac{1}{n+1} - \frac{1}{m+1} \to 0$

$\therefore |f_n|$ is Cauchy .

$\| f_n - f_0\| = \int_0^2 |f_n - f_0| = \int_0^1 (f_n - f_0) + \int_1^2 (f_n - f_0)$

$= \int_0^1 (x^n - 0) + \int_1^2 (1 - 1) = \int_0^1 x^n\,dx = \frac{1}{n+1} \to 0$

$\therefore f_n \to f_0$ but f_0 is not continuous in [0,2]

$\therefore$ f_n is divergent $\Rightarrow$ c[a,b] is in completes
$\therefore$ (x, < , >) is not Hilbert space .

Remark:-

Every inner product space is normed space but the converse is not necessary to be true .

Theorem:-

A norm on vector space is induced by an inner product if and only if it satisfies the parallelogram law.

Proof:-

To prove $<x,y> = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$ (Real case) is inner product which induces the norm.

1) $<x,x> = \frac{1}{4}(\|x+x\|^2 - \|x-x\|^2) = \frac{1}{4}(\|2x\|^2 - \|0\|^2)$
$= \|x\|^2 \geq 0 \quad \forall\ x \in X$
$\Rightarrow <x,x> > 0$ when $x \neq \theta$.

2) $<x,y> = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \frac{1}{4}(\|x+y\|^2 - \|-(y-x\|^2)$
$= \frac{1}{4}(\|y+x\|^2 - |-1|^2 . \|y-x\|^2) = \frac{1}{4}(\|y+x\|^2 - \|y-x\|^2)$
$= <y,x>$.

3) Let $\forall$ u,v,w $\in$ x

$<u+v,w> + <u-v,w> = 2<u,w>$ …….$*$

If $u=v \Rightarrow <2u,w> = 2<u,w>$

Now letting $u = \frac{1}{2}(x+y)$, $v = \frac{1}{2}(x-y)$ and w=z in $*$

We get $<x,z> + <y,z> = <u+v,w> + <u-v,w>$
$=2<u,w> = <2u,w> = <x+y,z>$.

4) <αx,y> = α<x,y> ∀α∈|R , x,y∈X (cheek)

∴ <x,y> is inner product which induces the norm

Then from above we can get.

Remark:-

Every Hilbert space is Banach space but the convers is not necessary to be true.

And we get the following Corollary.

Corollary:-

A Banach space is a Hilbert space if and only if the norm satisfies the parallelogram law.

Proof:-

⇒suppose x is Hilbert space

⇒x is Banach space (Remark)

⇒x is normed space (Remark)

⇒x satisfy the parallelogram law

Conversely ⇐ suppose x is Banach space set the norm satisfy the parallelogram law ⇒ x is inner product space (Theorem) ⇒ x is Hilbert space.

Example:-

Lp^n (is a Banach space under the norm)

$\|x\|_p = [\sum_{i=1}^{n} |x|^p]^{1/p}$ $1< p < \infty$ but not a Hilbert space , only in the case p=2

When we prove Hilbert space we must satisfy the parallelogram law

Let x= (1,1,0,0,…) and y= (1,-1,0,0,…)

x+y = (2,0,0,…) and x-y = (0,2,0,0,…)

We have

$\|x\| = [\sum_{i=1}^{n} |xi|^{p}]^{1/p} = (|1|^p + |1|^p + 0+0+\ldots.+0)^{1/p} = 2^{1/p}$

$\|y\| = [\sum_{i=1}^{n} |yi|^{p}]^{1/p} = (|1|^p + |-1|^p + 0+0+\ldots.+0)^{1/p} = 2^{1/p}$

$\|x+y\| = (|2|^p + 0+0+\ldots..+0)^{1/p} = (|2|^p)^{1/p} = 2$

$\|x-y\| = (0+|2|^p + 0+\ldots..+0)^{1/p} = (|2|^p)^{1/p} = 2$

The parallelogram law $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

$\Rightarrow (2)^2 + (2)^2 = 8$, $2(2^{1/p})^2 + 2(2^{1/p})^2$

Thus if p=2 the parallelogram law satisfy $\Rightarrow Lp^n$ is Hilbert space

if p≠2 the parallelogram law is not satisfied $\Rightarrow Lp^n$ is not Hilbert space.

Orthogonal Complements

Definition:-

1) Two vectors x and y in a Hilbert space X are called orthogonal , denoted by $x \perp y$ if $<x,y> = 0$.
2) A vector $x \in X$ is orthogonal to $\emptyset \neq A \in X$, denoted by $x \perp A$ if $<x,y> = 0 \quad \forall y \in A$.

3) Let $\emptyset \neq A \subset X$ then the set of all vector orthogonal to A denoted by $A^\perp$ is called the orthogonal complement of A

i.e $A^\perp = \{x \in X : <x,y> = 0 \ \ \forall y \in A\}$

or $A^\perp = \{x \in X : x \perp y \ \ \forall y \in A\}$($A^\perp$ real as A perpendicular).

Remark:-

$(A^\perp)^\perp$ will denoted orthogonal complement of $A^\perp$.

4) Two sets A and $B \subset X$ are called orthogonal denoted by $A \perp B$ if $<x,y> = 0 \quad \forall x \in A , y \in B$.

properties:-

1) The relation of orthogonality in a Hilbert space is symmetric i.e $x \perp y \Leftrightarrow y \perp x$.

proof:-

$x \perp y \Leftrightarrow <x,y> = 0 \Leftrightarrow \overline{<x,y>} = \bar{0}$

$\Leftrightarrow <y,x> = 0$ [since $\overline{<x,y>} = <y,x>$]

$\Leftrightarrow y \perp x$.

2) If $x \perp y \Rightarrow \alpha x \perp y$ for all α scalar

$x \perp y \Rightarrow \langle x,y\rangle = 0$

$\langle \alpha x,y\rangle = \alpha\langle x,y\rangle = \alpha .0 = 0 \Rightarrow \alpha x \perp y$.

3) $0 \perp x \;\; \forall x \in X$ since $\langle 0,x\rangle = 0 \;\; \forall x \in X$.

4) The zero vector is the only vector which is orthogonal to itself.

Proof:-

$x \perp x \Rightarrow \langle x,x\rangle = 0 \Rightarrow \|x\|^2 = 0 \Rightarrow x = 0$

hence if $x \perp x$ then x must be zero vector.

5) It is clear that $\{0\}^{\perp} = x$ and $x^{\perp} = \{0\}$.

Proof:- T.p $\{0\}^{\perp} = x$ let $x \in X$ since $\langle x,0\rangle = 0 \Rightarrow x \in \{0\}^{\perp}$

$\Rightarrow x \subset \{0\}^{\perp}$ but $\{0\}^{\perp} \subset x \Rightarrow \{0\}^{\perp} = x$

And T.p $x \in x^{\perp} \Rightarrow \langle x,y\rangle = 0 \;\; \forall y \in x$

Now take $x = y \Rightarrow \langle x,x\rangle = 0 \Rightarrow \|x\|^2 = 0 \Rightarrow x = 0 \Rightarrow x^{\perp} = \{0\}$.

6) It is clear that if $A \perp B$ then $A \cap B = \{0\}$.

Proof:-

Since $A \perp B \Rightarrow \langle x,y\rangle = 0 \;\; \forall x \in A$, $y \in B$

T.p $A \cap B = \{0\}$

Let $x \in A \cap B \Rightarrow x \in A$ and $x \in B \Rightarrow \langle x,x\rangle = 0$

$\Rightarrow \|x\|^2 = 0 \Rightarrow x = 0$

$\therefore A \cap B = \{0\}$.

Theorem :-

Let x be a Hilbert space and A be its arbitrary subset then the following result

1) $A^{\perp}$ is subspace of x .

2) $A \cap A^{\perp} \subset \{0\}$.

3) $A \cap A^{\perp} = \{0\} \Leftrightarrow$ A is subspace .

4) If $B \subset A$ then $A^{\perp} \subset B^{\perp}$.

5) $A \subset (A^{\perp})^{\perp}$.

Proof:-

1) Let $x,y \in A^{\perp}$ T.p $\alpha x+\beta y \in A^{\perp}$ for any scalar α,β

$\Rightarrow \langle x,z\rangle = 0 \;\; \forall z \in A$ and $\langle y,z\rangle = 0 \;\; \forall z \in A$

Thus for arbitrary scalar α,β we get

$\langle \alpha x+\beta y,z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle = \alpha.0+\beta.0=0$

$\therefore \alpha x+\beta y \in A^{\perp}$

$\therefore A^{\perp}$ is subspace of x.

2) Let $x \in A \cap A^{\perp} \Rightarrow x \in A$ and $x \in A^{\perp}$

$\therefore x \in A^{\perp} \Rightarrow x \perp y \;\; \forall y \in A \Rightarrow x \perp x$ [since $x \in A$]

$\Rightarrow \langle x,x\rangle = 0 \Rightarrow \|x\|^2 = 0 \Rightarrow x=0$

$\therefore A \cap A^{\perp} \subset \{0\}$.

3) $\Rightarrow$ suppose A is a subspace then $0 \in A$ and by 1) $A^{\perp}$ is subspace

$\Rightarrow 0 \in A^{\perp} \Rightarrow A \cap A^{\perp} = 0$.

4) Let $x \in A^{\perp} \Rightarrow \langle x,y\rangle = 0 \;\; \forall y \in A$

$\therefore \forall y \in B \Rightarrow y \in A$ $(B \subset A)$

Thus $\langle x,y\rangle = 0 \;\; \forall y \in B \Rightarrow x \in B^{\perp}$

$\therefore A^{\perp} \subset B^{\perp}$.

Theorem:- (By Thagorean theorem)

Let x be a Hilbert space if x orthogonal to y then

$\|x+y\|^2 = \|x\|^2 + \|y\|^2$

Proof:-

$\|x+y\|^2 = \langle x+y, x+y\rangle$

$= <x,x> + <x,y> + <y,x> + <y,y>$

Since x orthogonal to y $\Rightarrow <x,y> = 0$

also $\overline{<x,y>} = \bar{0} \Rightarrow <y,x> = 0$

$\therefore \|x+y\|^2 = <x,x> + 0 + 0 + <y,y> = \|x\|^2 + \|y\|^2$.

Definition:-

A set S of vector in Hilbert space is said to be orthogonal if

1) S is orthogonal .
2) $\|x\| = 1$ for every vector x in S .

Definition:-

A Seq [finite or infinite] of vectors x_n is called on orthogonal sequence if :-

1) $x_i \perp x_j \quad \forall i \neq j$.
2) $\|x_k\| = 1$ for all k

   i.e $<x_i, x_j> = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Example:-

1) In the unitary space c3 the vector

   $x_1 = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, $x_2 = (\frac{2}{3}, \frac{1}{3}, \frac{-2}{3})$, $x_3 = (\frac{2}{3}, \frac{-2}{3}, \frac{1}{3})$ are orthonormal.