Complex Analysis

References

Churchill R. V. " Complex Variables & Applications "

Chapter 1 : Complex Numbers

Definitions:

1) The complex number is an order pair (x, y).
2) $z = x + iy, \quad x + iy = 0 \quad iff \;\; x = 0 \;\; and \;\; y = 0.$
3) $x_1 + iy_1 = x_2 + iy_2 \quad iff \quad x_1 = x_2 \quad and \quad y_1 = y_2\,.$
4) $i = \sqrt{-1} \;\; , \quad i^2 = -1.$
5) $x = Rez \;\; , \; y = Imz \; , \; i \; is \; the \; imaginary \; numbers \; Unit\,.$
6) $The \; set \; of \; complex \; numbers \; denoted \; by \; \mathbb{C} \; and \quad R \subseteq \mathbb{C}.$

Operations on Complex Numbers

Let $z_1, z_2 \in C$ and $z_1 = x_1 + iy_1 \,, z_2 = x_2 + iy_2\,.\; Then$

1) $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \;\; \pm y_2\,).$
2) $z_1 . z_2 = (x_1 x_2 - y_1 \; y_2\,)i(x_1 y_2 \; + x_2 y_1\,).$
3) $\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} . \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{x_1 x_2 + y_1 \; y_2}{x_2{}^2 + y_2{}^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2{}^2 + y_2{}^2} \quad , z_2 \neq 0.$

Complex Conjugate

Definition : Let $z = x + iy$ be an complex number , then the conjugate 0f z denoted $\bar{z}$ Is $x - iy$, so $\bar{z} = x - iy$.

Ex . $If \;\; z = -3i + 9 + 2i \, , find \;\; \bar{z}? \quad z = 9 - i \quad \rightarrow \bar{z} = 9 + i$.

Note. Let $z \neq 0 \, , then \;\; \frac{1}{z} = \frac{1}{x + iy} . \frac{x - iy}{x - iy} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$.

Examples . Let $z_1 = 1 + 5i \;\; \& \, z_2 = -3 - 2i \quad , \; evalute$

1. $z_1 + z_2 \;\; = -2 + 3i$
2. $z_1 - z_2 = 4 + 7i$
3. $z_1 . z_2 = 7 - 17i$
4. $\frac{z_1}{z_2} = \frac{-13}{13} + i \frac{-13}{13} = -1 - i$
5. $\frac{1}{z_1} = \frac{1}{1 + i5} = \frac{1}{26} - i \frac{5}{26}$.

Properties of Complex Conjugate

1- If $z = 0 \rightarrow \bar{z} = 0$
2- $\bar{\bar{z}} = z$
3- $\bar{z} = z \ iff \ Imz = 0$
4- $\bar{z}.z = x^2 + y^2$
5- $\bar{z} + z = 2Rez = 2x$
6- $\bar{z} - z = 2Imz = 2iy$
7- $\overline{(z_1 \pm z_2)} = \overline{z_1} \pm \overline{z_2}$
8- $\overline{(z_1.z_2)} = \overline{z_1}.\overline{z_2}$
9- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \quad , z_2 \neq 0$.

Ex. Prove that

1. $\frac{\overline{(2+\iota)^2}}{3-4i} = 1$?
2. $z = i \pm 1$ are the roots of the equation $z^2 - 2zi - 2 = 0$?

Algebraic Properties

1- The comm. Law $z_1 + z_2 = z_2 + z_1 \quad , z_1.z_2 = z_2.z_1$
2- The asse. Law $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
3- The dist. Law $(z_1 + z_2).z_3 = z_1.z_3 + z_2.z_3$
4- Additive Identity $z + 0 = 0 + z = z$
5- Multiplication Identity $z.1 = 1.z = z$
6- Additive Inverse $(-z)$, $z + (-z) = -z + z = 0$
7- Multiplication Inverse $\left(\frac{1}{z}\right)$, $z.z^{-1} = z^{-1}.z = 1.$

Exercises :

$Q1$- Find the Re & Im parts for

$a)\ 5/(4i - 3) \quad , b)\ (3 + 4i)/(1 - i) \quad , c) \sqrt{\frac{1+i}{1-i}}$.

$Q2$–Write $\left(\frac{1}{2-3i}\right)\left(\frac{1}{1+i}\right)$ in complex form.

$Q3$ –Verify $\frac{5}{(1-i)(2-i)(3-i)} = \frac{1}{2}i$.

$Q4$– Solve the equation $z^2 + z + 1 = 0$.

$Q5 - Find\ the\ Multipl. Inverse\ for\ z = 3i - 2$.

Absolute Value of complex Number

Definition : The Modulus or Absolute Value of a complex number $z = x + iy$

defined as the nonnegative real number $\sqrt{x^2 + y^2}$ and denoted by $|z|$.

$\therefore z = \sqrt{x^2 + y^2}$.

Ex. $z = -3 - 4i \rightarrow |z| = \sqrt{9 + 16} = \sqrt{25} = 5.$

Some properties of $|z|$:

1- $|z| = \sqrt{z\bar{z}}$
2- $|z| = |\bar{z}|$
3- $|z_1 - z_2| = |z_2 - z_1|$
4- $\frac{1}{2}(z + \bar{z}) = x \leq \sqrt{x^2 + y^2} = |z|$
5- $\frac{1}{2}(z - \bar{z}) = y \leq \sqrt{x^2 + y^2} = |z|$
6- $|z_1 . z_2| = |z_1| . |z_2|$
7- $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$, $|z_2| \neq 0$
8- The Triangle Inequality $|z_1 + z_2| \leq |z_1| + |z_2|$, (proof).
9- $|z_1 + z_2| \geq ||z_1| - |z_2||$.

Q- Show that $\left|(2\bar{z} + 5)\left(\sqrt{2} - i\right)\right| = \sqrt{3}|2z + 5|$.

Geometric Representation of Complex Numbers

Definition **:** Each complex number z corresponds to just one point in complex plane and conversely .

Ex. $z = -3 + 2i \ \rightarrow x = -3\,, y = 2 \ \ , P(-3{,}2)$.

$|z_1 + z_2|$ (Fig.)

$OA + AB \geq OB$

$z_1 = \overrightarrow{OA}\ \ ,\ \ z_2 = \overrightarrow{OC} = \overrightarrow{AB}$

$\therefore |z_1| + |z_2| \geq |z_1 + z_2|$

$|z_1 + z_2| \leq |z_1| + |z_2|$

$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ (Fig.)

$|z| = r$, it's a circle with center $(0{,}0)$ $and\ \ radius\ \ r$.

$|z| = \sqrt{x^2 + y^2} = r \ \rightarrow x^2 + y^2 = r^2\ (The\ equation\ of\ circle\ )$

$|z - z_0| = r\ ,\ z_0 = x_0 + iy_0\,,$ it's a circle with center z_0 $and\ \ radius\ \ r$. (Fig.)

$|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2} = r \rightarrow (x - x_0)^2 + (y - y_0)^2 = r^2$.

$|z - z_0| \leq r$

$|z - z_0| \geq r$

$R_1 \leq |z - z_0| \leq R_2$

Ex. Graph: 1. $|z - 1 + i| = 1$, 2. $|z - 3i| < 2$.

Exercises:

Show that the following Graphs

$Q1.$ $|z + 2 - i| = 4$

$Q2.$ $|z + i| < 2,\ \ |z - i| > 3$

$Q3.$ $1 < |z - 3i| < 2$

$Q4.$ $2 < |z + 2 - i| \leq 3$

$Q5.$ $Re(z + 1) = 2\ \ ,\ \ \ \ Im(z - i) = 4.$

Polar Coordinates

Definition: Let $r(r > 0)$ and θ are the polar coordinates of the point $P(x, y)$ corresponding to a nonzero complex number $z = x + iy$,

since $x = r\cos\theta$ *and* $y = r\sin\theta$

$\therefore z = r(\cos\theta + i\sin\theta)$ polar form for z.

Notes: 1. The number r is the length of the vector z

$$\therefore r = |z| = \sqrt{x^2 + y^2}\ .$$

2. The number θ is called an **argument** of z or angle of the complex number z
   And written by $\theta = \arg z$, $\theta = \tan^{-1}\frac{y}{x}$.
3. The **principle value** of $\arg z$ denoted by $Arg\ z$ is defined as that unique value of $\mathrm{Arg}\, z$ such that $-\pi \leq Arg\ z < \pi$,
   $\therefore \arg z = Arg\ z + 2k\pi$, $k = 0, \pm 1, \pm 2, \dots..$
4. $\arg(z_1 z_2) = \arg z_1 + \arg z_2$.
5. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$, $z_2 \neq 0$. (H.W.)

Proof of 4:

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ & $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$
$$\therefore \arg(z_1 z_2) = (\theta_1 + \theta_2) = \arg z_1 + \arg z_2.$$

Remark: The property N0. 4 is always true , but this property for the principle value $(Arg z)$ Is not always true.

Ex. Take $z_1 = -1$, $z_2 = i$ $\rightarrow \theta_1 = \tan^{-1}\frac{0}{-1} = \pi$ $, \theta_2 = \tan^{-1}\frac{1}{0} = \frac{\pi}{2}$

$$z_1 z_2 = -i \ \rightarrow \theta_3 = \tan^{-1}\frac{-1}{0} = -\frac{\pi}{2}$$

$$\therefore Arg(z_1 z_2) \neq Arg z_1 + Arg z_2 \ \ ,, -\frac{\pi}{2} \neq \pi + \frac{\pi}{2}. \quad (-\pi \leq Arg z \leq \pi).$$

De Moivre's Theorem

Let n be an $+ve, -ve$ or $ractinal\ \ number$, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta)$.

Ex. 1. Use De Moivres Theorem to solve : $\frac{(\cos\theta - i\sin\theta)^3}{(\cos 2\theta + i\sin 2\theta)^2} = \frac{(\cos\theta + i\sin\theta)^{-3}}{(\cos\theta + i\sin\theta)^4} = (\cos\theta + i\sin\theta)^{-7}$.

Ex.2. Solve $(1 + i\sqrt{3}\,)^3 - (1 - i\sqrt{3}\,)^3$.

Solution : $\left(1 - i\sqrt{3}\,\right) = r(\cos\theta + i\sin\theta)$ $, r = \sqrt{1^2 + (\sqrt{3}\,)^2} = \sqrt{4} = 2$, $\theta = \tan^{-1}\frac{-\sqrt{3}}{1} = \frac{-\boldsymbol{\pi}}{\mathbf{3}}$,

$$\left(1 - i\sqrt{3}\,\right) = 2\ (\cos\tfrac{\pi}{3} - i\sin\tfrac{\pi}{3}).$$

$\therefore (1 - i\sqrt{3}\,)^3 = 2^3(cos\pi - isin\pi).$

$\left(1 + i\sqrt{3}\,\right) = 2(cos\frac{\pi}{3} + isin\frac{\pi}{3}).$

$\therefore (1 - i\sqrt{3}\,)^3 = 2^3(cos\pi + isin\pi).$

$\therefore (1 - i\sqrt{3}\,)^3 - (1 + i\sqrt{3}\,)^3 = 2^3(cos\pi + isin\pi - cos\pi + isin\pi) = 16isin\pi = 0.$

Euler's Formula

$e^{i\theta} = cos\theta + isin\theta$

$z = r(cos\theta + isin\theta) = re^{i\theta}$ Euler's formula

properties

1. $\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta},$
2. $z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}.$
3. $\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$

Ex. Write the number $1 + i$ in Euler's formula?

$r = \sqrt{2}\ \theta = \frac{\pi}{4} \rightarrow 1 + i = \sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right).$

Powers & Roots

1- **Powers:**

Let $z = re^{i\theta}, z \neq 0$,, $z^n = r^n e^{in\theta}$ $, n = 0, \pm 1, \pm 2, \dots \dots$

If $r = 1 \rightarrow z^n = e^{in\theta}$ & $(e^{i\theta})^n = (cos\theta + isin\theta)^n = cos\, n\theta + isin\, n\theta.$

Ex. Solve $(1 + i)^4$?

Let $(1 + i) = r(cos\theta + isin\theta) \quad \rightarrow r = \sqrt{2}\ ,\ \theta = \frac{\pi}{4}$

$\therefore (1 + i) = \sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)$

$\therefore (1 + i)^4 = 2^2(cos\pi + isin\pi).$

2- Roots :

Let $z^n = 1$, $z = re^{i\theta}$, $n = 0, \pm 1, \pm 2, \dots \dots$

Since $1 = 1.e^{i0}$

$\therefore (re^{i\theta})^n = 1.e^{i0} \rightarrow r^n = 1 \;\; and \;\; n\theta = 0 + 2k\pi$

$\therefore \theta = \frac{2k\pi}{n}$, $k = 0,1,2,3, \dots \dots$

$\therefore z = e^{\frac{i2k\pi}{n}} = \cos\frac{2k\pi}{n} + isin\frac{2k\pi}{n}$, the set of roots for 1.

$Now \;\; let \;\; w = \rho(cos\emptyset + isin\emptyset), \quad w \in \mathbb{C} \;\&\; w \neq 0.$

$\sqrt[n]{w} = \sqrt[n]{\rho}\,(cos\frac{\emptyset+2k\pi}{n} + isin\frac{\emptyset+2k\pi}{n})$, the general law to solve roots $\forall z \in \mathbb{C}$.

Ex. Find all roots for $\sqrt[3]{\sqrt{6} - \sqrt{2}\, i}$?

Solution : $\rho = \sqrt{6+2} = 2\sqrt{2}$, $\emptyset = tan^{-1}\frac{-\sqrt{2}}{\sqrt{6}} = tan^{-1}\frac{-1}{\sqrt{3}} = \frac{-\pi}{6}$. $n = 3$, $k = 0,1,2$.

$If \;\; k = 0 \;\rightarrow w_0 = \sqrt[3]{2\sqrt{2}}\left(cos\frac{\pi}{18} - isin\frac{\pi}{18}\right) = \sqrt{2}\left(cos\frac{\pi}{18} - isin\frac{\pi}{18}\right).$

$If \;\; k = 1 \rightarrow w_1 = \sqrt{2}\left(cos\frac{11\pi}{18} - isin\frac{11\pi}{18}\right).$

$If \;\; k = 2 \rightarrow w_2 = \sqrt{2}\left(cos\frac{23\pi}{18} - isin\frac{23\pi}{18}\right).$

Exercises:

1- Find all roots for: ***a***) $(2i)^{\frac{1}{2}}$, ***b***) $(-i)^{\frac{1}{3}}$, ***c***) $\sqrt[4]{-i}$.

2- By using the polar form prove that :

  a) $i(1 - i\sqrt{3})(\sqrt{3} + i) = 2 + i\sqrt{3}.$ ***b***) $(-1+i)^7 = -8(1+i)$.

3- $if \;\; z^2 - 2zcos\theta = 1 = 0 \;\; solve$ ***a***) $z^m + \frac{1}{z^m}$, ***b***) $z^m - \frac{1}{z^m}$.

3- Express in polar form : ***a***) $-2\sqrt{2} - 2\sqrt{2}\, i$, ***b***) $-1 + \sqrt{3}\, i$.

4- Find all values of : ***a***) $z^5 = 1$, ***b***) $z^4 = i$.

5- Let $z = \rho e^{i\emptyset}$, $prove \; that \; \left|e^{iz}\right| = e^{-\rho sin\emptyset}$?

Regions in Complex Plane

Def.1. Let $\mathbb{C}$ be the set of all complex numbers, let $z_0 \in \mathbb{C}$ and $\in$ be any positive real number . We say that the set $N(z_0, \in)$ is the **Neighborhood** of z_0 and Written by

$$\{z \in \mathbb{C}: \; |z - z_0| < \in.\}$$

Def.2. Let S be a set of all complex numbers and let z_0 be any complex number , then :

1- z_0 is said to be **Interior point** of $S \; iff \; \exists \, an \, \in -neighbarhood$ which is inside

$N(z_0, \in) < S$.

2- z_0 is said to be **Exterior point** of S iff $\exists\, N(z_0, \in)$ which contains No point 0f S, that's means $N(z_0, \in) \cap S = 0$.

3- z_0 is said to be **Boundary point** if each $N(z_0, \in)$ contains points both in S and not in S.

Ex. $|z| = 1$ it is a circle with center $(0,0)$ and radius $r = 1$, so
$|z| < 1$ is interior points, $|z| > 1$ is exterior points and $|z| = 1$ is boundary points.

Def.3. A set is said to be **Open Set** if each of its points is an interior point.

Def.4. A set is said to be **Closed Set** if it is contains all of its boundary points.
Def.5. The **Closure** $\bar{S}$ of S is the set $S \cup \text{boundary points.}$

Ex. The set $S_1 = \{z: 0 < |z| < 1\}$ is open set;
The set $S_2 = \{z: 0 \le |z| \le 1\}$ is open set;
The set $S_3 = \{z: 0 < |z| \le 1\}$ is neither open nor closed.

Def.6. An open set S is **Connected** if each pair of points in it can joined by polygonal path, Consisting of a finite number of line-segments joined end to end which lies entirely in S.

Ex. The open set $|z| < 1$ is connected; the ring $1 < |z| < 2$ is connected;
The set $\{z : \ |z| < 1 \ \text{ or } \ |z| > 2\}$ is not connected.

Def.7. By a **Domain** we mean an open set which connected.
Def.8. By a **Region** we mean a domain together with some ,none ,or all its boundary points.
Def.9. A set S is **Bounded** if every point in S lies inside some circle $|z| = R$, otherwise it is **Unbounded**.

Def.9. A point z_0 is said to be an **Accumulation point (limit point)** of a set S if each $N(z_0, \in)$ Contains at least one point of S distinct from z_0.

Ex. Let the set $S = \left\{\frac{i}{n}\right\}$, $n == 1,2,3, \ldots.$
$\therefore S = \{i, \frac{1}{2}i, \frac{1}{3}i, \frac{1}{4}i, \ldots.\}$,

1- what's the interior points & the boundary points for the set S? no interior points, but all the points of this set are boundary.
2- the set S have an limit point? yes, $z_0 = 0$, $it\ is\ unique$.

3- does the set S boundary? Yes.
4- Does the set S closed ?
5- 5- Does the set S open ?
6- 6- Does the set S connected ?
7- 7- Does the set S a domain ?

what's the closure of the set S ? $\bar{S} = \{0, i, \frac{1}{2}i, \frac{1}{3}i, \frac{1}{4}i, \dots.\}$