Chapter Four

Electrostatic field in dielectric media

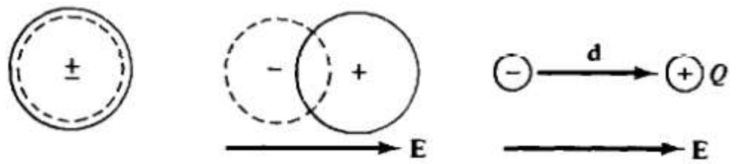
Text, ch.4
Griffith's, ch.4
Jackson, ch.3
محاضرات د.نعمة الدليمي
UIUC Physics 435 EM Fields & Sources I Fall Semester, 2007 Lecture Notes 10 Prof. Steven Errede
Schaum's ch.7

4– 1: Polarization الاستقطاب في العوازل

كما درسنا في الفصل الثاني, فان المواد تصنف الى ثلاث انواع هي الموصلات والعوازل واشباه موصلات. تم تناول بعض المفاهيم (مثل المجال الكهربائي والجهد غير المتجة وقانون كاوس) للاوساط الموصلة في الفصول السابقة, اما في هذا الفصل فسوف نتناول نفس المفاهيم ولكن في الاوساط العازلة.

وفقا لنظرية التركيب الذري, فأن ذرات المادة الموصلة تمتلك الكترونات لاترتبط بها بشكل خاص انما تشترك كل الذرات بكل الالكترونات لذلك تسمى الكترونات الموصلات بـ"السحابة الالكترونية.

اما في الاوساط العازلة, فأن الوضع يختلف، حيث الالكترونات تختص بكل ذرة وترتبط بها بشكل قوي، اي هي مقيدة بشده بتلك الذرات. ان كل ما تستطيع فعله هذه الشحنات هو التحرك بشكل محدود جدا داخل الذره او الجزيئة في حال تسليط مجال كهربائي خارجي عليها، ان هذا النوع من الازاحة يكون ضئيل او "مايكروي (microscopic)". تدعى ظاهرة ازاحة مركزي الشحنات الموجبة (النوى) والسالبة (الالكترونات) بالاستقطاب Polarization, P، كما في الشكل ادناه.



(a) (b) (c)

وللتذكير فأن الاستقطاب هنا يخص "اعادة ترتيب شحنات الوسط العازل" اما في البصريات فهو عملية تخص انتظام اهتزازية الموجات بالنسبه الى اتجاه انتشار (تقدم) الموجة.

ان ازاحة الالكترونات والذرات في المادة العازله تسلك ما يشبه عملية حركة نابض يربط بين كتلتين في الميكانيك الكلاسيكي، والتي تخضع لقانون هوك Hook's law ، $\vec{F} = -kx$ في حالة الازاحة ببعد واحد، حيث F هي القوة المعيدة restoring force (قوة المرونة) والتي

تتناسب طرديا مع الازاحة x. ان قوة المرونة في الكهربائية الساكنة تعالج حالة القوة بين شحنتين في وسط عازل بدلا من القوة بين كتلتين في الميكانيك الكلاسيكي.

ان قوة المرونة تسلك عكس سلوك قوة كولوم والتي تتناسب عكسيا مع مربع الازاحة، اي ان قوة كولوم تقل بزيادة الازاحة بين مركزي النوعين من الشحنات.

عند تسليط مجال كهربائي $\vec{E}$ فان الشحنات الموجبة تزاح بنفس اتجاة المجال المسلط, بينما تزاح الشحنات السالبة بعكس اتجاة المجال. على فرض ان مقدار الازاحة هو d فان العزم المتولد والمسمى بعزم ثنائي القطب ببعد واحد يعطى:

$$\text{Electric dipole moment:} \quad \vec{p} = q\,x \;.\;.\;. \;(4-1)$$

حيث q الشحنة و x الازاحة.

لأغلب العوازل فان مركزي الشحنتين الموجبة والسالبة يعود ليتطابق بمجرد زوال المجال المسلط.

4 – 2: Electrostatic field produced outside a polarized object, $\vec{E}_p$

مجال الكهربائية الساكنة المنتج بواسطة عازل مستقطب – لنقطة تقع خارج الوسط

Consider a finite piece of dielectric material which is polarized. Each volume element $\Delta\acute{v}$ of the dielectric medium is characterized by a dipole moment $\Delta p = \vec{P}(\acute{r})\Delta\acute{v}$, where $\vec{P}(\acute{r})$ is the total polarization vector or dipole moment per unit volume. Since the distance between observing point $p(x,y,z)$ and $\Delta\acute{v}$ is large compared with the dimensions of $\Delta\acute{v}$, some mathematical approach can be carried out, this quantity (dipole moment) completely determines $\Delta\acute{v}$'s contribution to the potential:

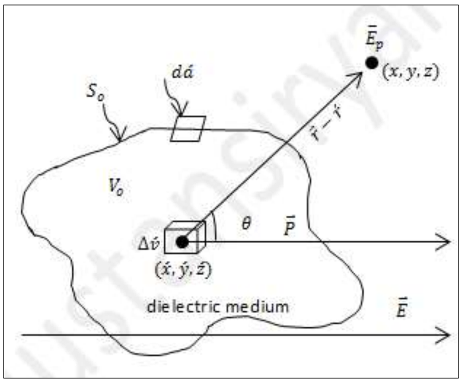
$$\Delta U_p(\vec{r}) = \frac{\Delta\vec{p}.(\vec{r}-\acute{\vec{r}})}{4\pi\epsilon_o|\vec{r}-\acute{r}|^3} = \frac{\vec{P}(\acute{r}).(\vec{r}-\acute{\vec{r}})\Delta\acute{v}}{4\pi\epsilon_o|\vec{r}-\acute{r}|^3} \quad .\;.\;.\;(1)$$

where we know that, using in special case of eq. (2-14), the potential for a single dipole is;

$$U_p(\vec{r}) = \frac{+q}{4\pi\epsilon_o|\vec{r}-\acute{r}_1|} + \frac{-q}{4\pi\epsilon_o|\vec{r}-\acute{r}_2|}$$

$$= \frac{\vec{P}(\acute{r}).\hat{u}_r}{4\pi\epsilon_o|\vec{r}-\acute{r}_1|^2}$$

where $\hat{u}_r$ is the unite vector directed from the dipole toward the point p.



Integrating for all medium space in order to compute the potential due to all dipoles, we get;

الجهد الكلي (خارج المادة) المتولد بتأثير كل ثنائيات الاقطاب الموجوده في المادة

$$U_p(\vec{r}) = \frac{1}{4\pi\epsilon_o}\int_{V_o} \frac{\vec{P}(\acute{r}).(\vec{r}-\acute{\vec{r}})d\acute{v}}{4\pi\epsilon_o|\vec{r}-\acute{r}|^3} \quad . . . (2)$$

There is a directional relation between primed and unprimed coordinates in our equation; $\vec{\nabla} = -\acute{\vec{\nabla}}$, thus; the following identity is changed to be;

$$\acute{\vec{\nabla}}\left(\frac{1}{|\vec{r}-\acute{r}|}\right) = +\frac{\vec{r}-\acute{r}}{|\vec{r}-\acute{r}|^3} \quad . . . (3)$$

Thus;

$$\frac{\vec{P}(\acute{r}).(\vec{r}-\acute{\vec{r}})}{|\vec{r}-\acute{r}|^3} = \vec{P}(\acute{r}).\acute{\vec{\nabla}}\left(\frac{1}{|\vec{r}-\acute{r}|}\right) \quad . . . (4)$$

Substituting last eq. in eq.(2), yields;

$$U_p(\vec{r}) = \frac{1}{4\pi\epsilon_o}\int_{V_o} \vec{P}(\acute{r}).\left(\acute{\vec{\nabla}}\left(\frac{1}{|\vec{r}-\acute{r}|}\right)\right)d\acute{v} \quad . . . (5)$$

Eq.(5) can be further transformed by means of the vector identity (6) in table (1-1):

$$\acute{\vec{\nabla}}.\left(\varphi\vec{F}\right) = \varphi\acute{\vec{\nabla}}.\vec{F} + \vec{F}.\acute{\vec{\nabla}}\varphi$$

Let: $\varphi = \frac{1}{|\vec{r}-\acute{r}|}$ and $\vec{F} = \vec{P}(\acute{r})$, then:

$$\acute{\vec{\nabla}}.\left(\frac{\vec{P}(\acute{r})}{|\vec{r}-\acute{r}|}\right) = \frac{1}{|\vec{r}-\acute{r}|}\acute{\vec{\nabla}}.\vec{P}(\acute{r}) + \vec{P}(\acute{r}).\acute{\vec{\nabla}}\left(\frac{1}{|\vec{r}-\acute{r}|}\right)$$

$$\rightarrow \qquad \vec{P}(\acute{r}).\vec{\acute{\nabla}}\left(\frac{1}{|\vec{r}-\acute{r}|}\right)=\vec{\acute{\nabla}}.\left(\frac{\vec{P}(\acute{r})}{|\vec{r}-\acute{r}|}\right)-\frac{\vec{\acute{\nabla}}.\vec{P}(\acute{r})}{|\vec{r}-\acute{r}|} \quad ...(6)$$

Sub. last eq. in eq.(5), yields;

$$U_p(\vec{r})=\frac{1}{4\pi\epsilon_o}\left[\int_{V_o}\vec{\acute{\nabla}}.\left(\frac{\vec{P}(\acute{r})}{|\vec{r}-\acute{r}|}\right)+\int_{V_o}\frac{-\vec{\acute{\nabla}}.\vec{P}(\acute{r})}{|\vec{r}-\acute{r}|}d\acute{v}\right] \quad ... \ (7)$$

Appling the divergence theorem $\left(\int_{V_o}\vec{\acute{\nabla}}.\vec{F}d\acute{v}=\oint_{S_o}\vec{F}.\hat{n}d\acute{a}\right)$ in the 1st term from the last eq., yields;

$$\int_{V_o}\vec{\acute{\nabla}}.\left(\frac{\vec{P}(\acute{r})}{|\vec{r}-\acute{r}|}\right)=\oint_{S_o}\vec{P}(\acute{r}).\hat{n}d\acute{a}$$

Sub. this eq. into eq.(7), yields;

$$U_p(\vec{r})=\frac{1}{4\pi\epsilon_o}\left[\oint_{S_o}\frac{\vec{P}(\acute{r}).\hat{n}d\acute{a}}{|\vec{r}-\acute{r}|}+\int_{V_o}\frac{-\vec{\acute{\nabla}}.\vec{P}(\acute{r})}{|\vec{r}-\acute{r}|}d\acute{v}\right] \quad ... \ (8)$$

The two quantities $\vec{P}.\hat{n}$ and $-\vec{\acute{\nabla}}.\vec{P}$ which appeared in last equation are two scalar quantities obtained from the polarization $\vec{P}$, such that;

$$\left.\begin{aligned}\sigma_p&=\vec{P}.\hat{n}=\vec{P}_n\\ \rho_p&=-\vec{\acute{\nabla}}.\vec{P}\end{aligned}\right\} \quad .... \quad (9)$$

which are the polarization charge densities for the medium and it can be re-writing as: $Q_p=\oint_{S_o}\vec{P}(\acute{r}).\hat{n}d\acute{a}+\int_{V_o}-\vec{\acute{\nabla}}.\vec{P}(\acute{r})\,d\acute{v}$ and then:

$$U_p(\vec{r})=\frac{1}{4\pi\epsilon_o}\int\frac{dQ_p}{|\vec{r}-\acute{r}|}$$

ان كثافة شحنة الاستقطاب تعبر عن الشحنات غير حرة الحركة داخل المادة لذا فأنه يطلق عليها احيانا بالشحنات المقيدة (bounded). بالنسبة الى النوع السطحي σ_p فانها تقيس كثافة الشحنات لمركبة الاستقطاب العمودية على السطح S_o، بينما النوع الحجمي ρ_p فهو يقيس كثافة الشحنات داخل الوسط.

Substitution of eq.(9) in eq.(8) yields:

$$U_p(\vec{r}) = \frac{1}{4\pi\epsilon_o}\left[\oint_{S_o} \frac{\sigma_p d\acute{a}}{|\vec{r}-\acute{r}|} + \int_{V_o} \frac{\rho_p d\acute{v}}{|\vec{r}-\acute{r}|}\right] \quad . . . (8)$$

Since U is a function of the coordinates (x, y, z), to calculate the field $\vec{E}_p(\vec{r}) = -\vec{\nabla}U_p(\vec{r})$, the appropriate gradient is the $(-\vec{\nabla})$.

$$\vec{E}_p(\vec{r}) = -\vec{\nabla}\left\{\frac{1}{4\pi\epsilon_o}\left[\oint_{S_o} \frac{\sigma_p d\acute{a}}{|\vec{r}-\acute{r}|} + \int_{V_o} \frac{\rho_p d\acute{v}}{|\vec{r}-\acute{r}|}\right]\right\}$$

Noting that $\vec{\nabla}\left(\frac{1}{|\vec{r}-\acute{r}|}\right) = -\frac{\vec{r}-\acute{r}}{|\vec{r}-\acute{r}|^3}$ and again using equation (3) we get;

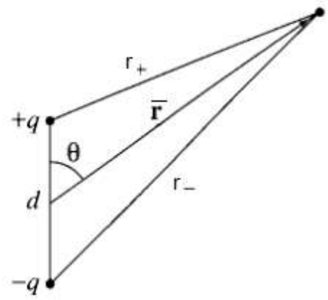
$$\boxed{\rightarrow \vec{E}_p(\vec{r}) = \frac{1}{4\pi\epsilon_o}\left[\oint_{S_o} \frac{\sigma_p(\vec{r}-\acute{r})d\acute{a}}{|\vec{r}-\acute{r}|^3} + \int_{V_o} \frac{\rho_p(\vec{r}-\acute{r})d\acute{v}}{|\vec{r}-\acute{r}|^3}\right]}$$

Example: (Jackson 3.10)

A physical dipole moment of two equal and opposite charges $(\pm q)$ separated by a distance (d), find the approximate potential at a distance far away from the dipole.

Solution:

let r_- be the distance from $(-q)$ and r_+ is the distance from $(+q)$, such that the setup will be as in the following fig.;



then, the potential due to these two charges is;

$$U_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_o}\left(\frac{+q}{r_-} + \frac{-q}{r_+}\right)$$

$$= \frac{q}{4\pi\epsilon_o}\left(\frac{1}{r_-} - \frac{1}{r_+}\right)$$

Because of $\vec{r} \gg d$, we can assuming that $\vec{r}_- \parallel \vec{r}_+$. Appling the law of cosines:

$$r_-^{\,2}(r_+^{\,2}) = r^2 + (d/2)^2 \pm rdcos\theta$$

$$= r^2\left(1 \pm \frac{d}{r}cos\theta + \frac{d^2}{4r^2}\right)$$

Because of $\vec{r} \gg d$, the 3ed term negligible, and the binomial expansion yields: $\frac{1}{r_\pm} \cong \frac{1}{r}\left(1 \pm \frac{d}{r}cos\theta\right)^{-\frac{1}{2}} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}cos\theta\right)$

Thus; $\frac{1}{r_-} - \frac{1}{r_+} \cong \frac{d}{r^2}cos\theta$

$$\rightarrow \quad U_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_o}\frac{qdcos\theta}{r^2} = \frac{1}{4\pi\epsilon_o}\frac{pcos\theta}{r^2}$$

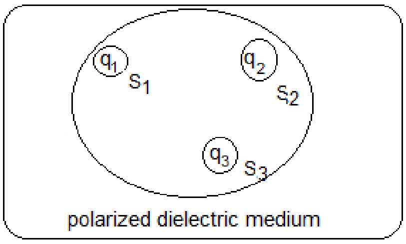
or in vector notation: $d\cos\theta = \hat{r}.\vec{d}$, thus:

$$U_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_o}\frac{p}{r^2}\hat{r}$$

في الواقع فأن الجهد الناتج في هذا المثال، بتأثير ثنائي قطب dipole، يتناسب مع $\frac{1}{r^2}$، في حين انه يتناسب مع $\frac{1}{r}$ في حالة احادي القطب monopole، ومع $\frac{1}{r^3}$ في حالة رباعي القطب Quadrupole، ومع $\frac{1}{r^4}$ في حالة ثماني القطب octopole.

<u>4 – 3: Gauss's law in dielectrics</u> قانون كاوس في الاوساط العازلة

In the following fig., the entire surface S is an imaginary closed surface located inside a dielectric medium. There is a certain amount of free charge, Q_{free}, in the volume V enclosed by S, and we shall assume that this free charge exists on the surface of three conductors in amounts q_1, q_2, and q_3. By Gauss law;



$$\oint_S \vec{E}.\vec{n}\,da = \frac{1}{\epsilon_\circ}\left(Q_{free} + Q_p\right) \quad ...\ (1)$$

where:
S: is the surface bounding the volume V.
Q_{free}: free charges inside conductors S_1, S_2 ,S_3.
Q_p: polarized charges inside the dielectric and onto its surface, such that:

$$Q_p = -\int_V (\vec{\nabla}.\vec{P})\ dV + \oint_{S_1+S_2+S_3} (\vec{P}.\hat{n})da$$

Appling the div. theorem in the volume integral term in last eq., yields;

$$Q_p = -\oint_{S+S_1+S_2+S_3} (\vec{P}.\hat{n})da + \oint_{S_1+S_2+S_3} (\vec{P}.\hat{n})da = -\oint_S (\vec{P}.\hat{n})da$$

sub. in eq.(1);

$$\oint_S \vec{E}.\vec{n}\,da = \frac{1}{\epsilon_\circ}\left(Q_{free} - \oint_S (\vec{P}.\hat{n})da\right)$$

$$\boxed{\oint_S (\epsilon_\circ\vec{E} + \vec{P}).\vec{n}\,da = Q_{free}}$$

Integral form of Gauss's low in dielectric

Inside the closed surface, the vector summation$\left(\epsilon_\circ\vec{E}+\vec{P}\right)$ may be replaced by a vector $\vec{D}$, and the above equation can be re-writing as;

$$\oint_s \vec{D}\,.\vec{n}\,da = Q_{free}$$

where $\vec{D}$ is called the *electric displacement vector*
also, $Q_{free}=\rho dV$, and again applying the div. theorem, yields:

$$\boxed{\vec{\nabla}.\vec{D}=\rho}$$

Differential form of Gauss's low in dielectric

For isotropic (homogeneous) dielectric media, the polarization vector $\vec{P}$ and the electrostatic field $\vec{E}$ have the same direction. For a linear medium $\vec{P}$ is directly proportional to $\vec{E}$, thus we can write;

$$\vec{P}\propto\vec{E}$$

$$\boxed{\vec{P}=\chi\vec{E}}$$

where χ is the *electric susceptibility* constant, which is a dimensionless constant.

But: $\vec{D}=\epsilon_\circ\vec{E}+x\vec{E}$

$$=(\epsilon_\circ+x)\vec{E} \quad ... \quad (\#)$$

If $x=0$,i.e. the medium i

s a vacuum , and then :

$$\boxed{\vec{D}_\circ=\epsilon_\circ\vec{E}} \quad \text{(electric displacement for vacuum)}$$

otherwise: $\boxed{\vec{D}=\epsilon\vec{E}}$ (electric displacement for medium)

where, ϵ is the Permittivity is a measure of the ability of a dielectric material to be polarized by an electric field. Sub. last equation in left hand side of eq.(#), yields;

$$\varepsilon\vec{E}=\varepsilon_\circ\left(1+\frac{x}{\epsilon_\circ}\right)\vec{E}\rightarrow\varepsilon=\varepsilon_\circ\left(1+\frac{x}{\varepsilon_\circ}\right)$$

$\rightarrow$ *kappa* or $\boxed{K=\varepsilon_r=\frac{\varepsilon}{\varepsilon_\circ}=1+\frac{x}{\varepsilon_o}}$

K is the *dielectric constant* (relative permittivity) of a medium, which is also pure number (dimensionless) quantity.

The dielectric constant (k) of a material is the ratio of its permittivity ε to the permittivity of vacuum εo, so k = ε/εo. The

dielectric constant is therefore also known as the relative permittivity of the material. Since the dielectric constant is just a ratio of two similar quantities, it is dimensionless.
Given its definition, the dielectric constant of vacuum is 1. Any material is able to polarize more than vacuum, so the k of a material is always > 1. Note that the dielectric constant is also a function of frequency in some materials, e.g., polymers, primarily because polarization is affected by frequency.
A low-k dielectric is a dielectric that has a low permittivity, or low ability to polarize and hold charge. Low-k dielectrics are very good insulators for isolating signal-carrying conductors from each other. Thus, low-k dielectrics are a necessity in very dense multi-layered IC's, wherein coupling between very close metal lines need to be suppressed to prevent a degradation in device performance.
A high-k dielectric, on the other hand, has a high permittivity. Because high-k dielectrics are good at holding charge, they are the preferred dielectric for capacitors. High-k dielectrics are also used in memory cells that store digital data in the form of charge.

4 – 4: Electrostatic field produced inside a polarized object, Lorentz field model $\vec{E}_s$

ان العازل يكتسب الاستقطاب نتيجة لاعادة توزيع الشحنات داخله بتأثير المجال المسلط.

من وجهة نظر جاهرية macroscopic، فأن صافي الشحنة داخل الوسط المستقطب يكون صفر بالرغم من اعادة اصطفاف الشحنات سالبه على اقصى يمين المادة والموجبه على الطرف الايسر، بافتراض اتجاة المجال المسلط هو من اليسار الى اليمين. ان فكرة كون صافي الشحنة صفر داخل الوسط العازل يمكن ان تفهم فقط في حال افتراض ان عنصر التفاضل الحجمي هو جاهري بحيث ان الوسط يتكون من كمية متساويه من الشحنات الموجبة والسالبة والنتيجة فأن صافي الشحنة صفر.

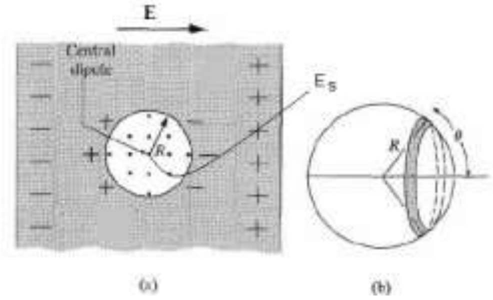
الشكل اعلاه يوضح الفكرة ببعد واحد، حيث ان الشحنة في نهاية كل ثنائي قطب تلغي بداية الشحنة المجاورة لها، بالنتيجة ستبقى شحنتيه متعاكستين في الاشارة في كل نهاية من السلسلة اعلاه، هذه الشحنات السطحية هي شحنات مقيدة.

هنالك اكثر من تصور اقترح لحل مسألة حساب مجال الكهربائية الساكنة داخل الوسط العازل المستقطب، اي مجال ثنائيات الاقطاب المتولدة بتأثير مجال كهربائي خارجي مسلط.

احد النماذج اقترح حساب المجال الكهربائي لكرة متناهيه في الصغر ضمن العازل مع استبعاد تأثير المجال الناتج من الشحنات خارج الكرة.

بينما اقترح نموذج اخر وجود سطح كروي فارغ داخل الوسط العازل ويتم حساب المجال في مركز الكرة نتيجة تأثير الشحنات خارج الكرة.

In order to determine the electrostatic field at the center of a spherical cavity inside a polarized medium and due to the surface polarized charges on its surface, as in the following fig.;



From last discussion, the distribution of charges is only on sphere's surface. According to Coulomb's law, the electric field is given as;

$$\vec{E}_{sph}(\vec{r}) = \frac{1}{4\pi\epsilon_o}\int_0^{\pi} \frac{\vec{R}}{\left|\vec{R}\right|^3}\sigma_p(r)da \quad . . . \quad (1)$$

from eq.(?), we have: $\sigma_p = \vec{P}.\hat{n} = P\cos(\pi - \theta) = -Pcos\theta$, also in spherical coor.: $da = r^2 sin\theta d\theta d\varphi$, substituting these in eq.(1), yields;

$$\rightarrow \quad \vec{E}_{sph}(\vec{r}) = \frac{1}{4\pi\epsilon_o}\int_0^{\pi}(-Pcos\theta)\frac{\vec{R}}{\left|\vec{R}\right|^3}(cos\theta)R^2 sin\theta d\theta d\varphi$$

the additional $cos\theta$ term is added because we are evaluating the field that only is in the polarization direction.

$$\vec{E}_{sph}(\vec{r}) = \frac{P}{4\pi\epsilon_o}\int_0^{\pi} cos^2\theta(-sin\theta)d\theta \int_0^{2\pi} d\varphi \ \hat{R}$$

$$= \frac{P}{2\epsilon_o}\left\{\frac{cos^3\theta}{3}\bigg|_0^{\pi}\right\}\hat{R}$$

$$= \frac{P}{2\epsilon_o}\frac{1}{3}[-1-(+1)]$$

$$\boxed{\rightarrow \vec{E}_{sph}(\vec{r}) = \frac{-\vec{P}}{3\epsilon_o}}$$

This is the macroscopically averaged E -field at the center of an imaginary small diameter sphere of radius R (somewhere) deep inside of a uniformly polarized dielectric [Prof. Steven Errede, د.نعمة].

4 – 5: Point charge in a dielectric fluid:

Assume us apply Gauss's law to a spherical surface of radius r encloses some electric charge q which is located inside a dielectric fluid, such that it behaves as a linear, isotropic medium. In this situation, $\vec{E}$, $\vec{D}$ and $\vec{P}$ are all parallel. For convenient, q will be located at the origin. Then, application of Gauss's law, in scalar form;

$$AD = q_{free}$$

$$4\pi r^2 D = q_{free}$$

$$D = \frac{q_{free}}{4\pi r^2}$$

$$\vec{D} = \frac{q_{free}}{4\pi r^3}\hat{r} \qquad vector\ form$$

we have:

$\vec{\mathrm{P}} = \chi\vec{\mathrm{E}}$, $\mathrm{K} = \frac{\epsilon}{\epsilon_o}, \chi = \epsilon_o\mathrm{K} - 1$, then;

$$\vec{E} = \frac{q}{4\pi \mathrm{K}\epsilon r^3}\hat{r}$$

also;

$$\boxed{\vec{P} = \frac{(\epsilon_o\mathrm{K} - 1)q}{4\pi \mathrm{K}\epsilon r^3}\hat{r}}$$

تشير المعادلة الاخيرة الى انه ونظرا للعلاقة $\vec{P} = \chi\vec{E}$ ، فأن المجال الكهربائي في حالة وجود وسط عازل هو اصغر بمقدار K عن المجال بغياب ذلك الوسط. حيث ان المجال داخل العازل يتولد من حاصل جمع المجال الذي تولده الشحنات المقيدة زائدا الحرة. عدد الشحنات الحرة هو واحد وهو q بينما الشحنات المقيدة تتألف من نوعين كما نعلم وهما: داخل الوسط ويتمثل بـ $\rho_p = \vec{\nabla}.\vec{P}$ وعلى السطح (بالتماس مع الشحنه) ويتمثل بـ σ_p. لو ادخلنا مؤثر $\vec{\nabla}.$ على المعادلة الاخيرة لكان الناتج صفر، اي لن توجد شحنات حجمية مستقطبة. عليه فأن محصله الشحنات المستقطبة في هذه الحالة سيكون كالاتي:

$$\sigma_p = \lim_{b\to o} 4\pi b^2 \left(\vec{P}.\hat{n}\right)_{r=b} = -\frac{(\mathrm{K} - 1)q}{\mathrm{K}}$$

Thus, the total charge is;

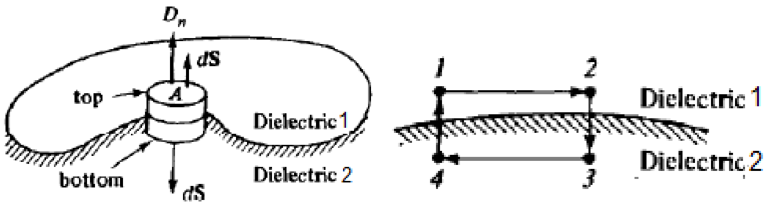
$$\sigma_p + q = \frac{1}{\mathrm{K}} q$$

$$\boxed{\therefore \sigma_p = q(\frac{1}{\mathrm{K}} - 1)}$$

4 – 6: Boundary conditions on the field vectors:

Schaum, p.85 & p.100. Q.7 in text.

In order to treat the problem of electrostatic field in an interface between two media, which involves; dielectric-conductor, dielectric-vacuum, and finally dielectric-dielectric, suppose the following argument.



سنتناول هنا حالة عامة لحساب كثافة الفيض والمجال الكهربائي والجهد في منطقة الحد الفاصل بين وسطين لهما خواص مختلفة وللمركبتين العمودية والموازية للحد الفاصل

لنرسم سطح افتراضي اسطواني S يقع ضمن الوسطين، ارتفاعه صغير مقارنة بقطر قاعدته، ويقطع عنصر تفاضلي من مساحة الحد الفاصل مقداره ΔS.

The free charges enclosed by the surface S are;

$$\sigma\, \Delta S + \frac{1}{2}(\rho_1 + \rho_2) \times volume$$

But $volume$ negligibly small, applying Gauss's law to S;

$$\vec{D}_2 . \hat{n}_2 \Delta S + \vec{D}_1 . \hat{n}_1 \Delta S = \sigma\, \Delta S$$

but $\hat{n}_1$ and $\hat{n}_2$ are ant direction; thus we can write:

$$\vec{D}_2 . \hat{n}_2 \Delta S + \vec{D}_1 . (-\hat{n}_2) \Delta S = \sigma\, \Delta S$$

$$(\vec{D}_2 - \vec{D}_1) . \hat{n}_2 = \sigma$$

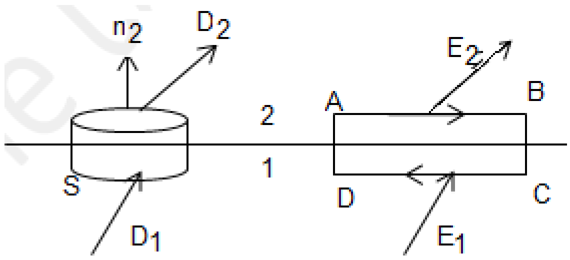
let $\hat{n}_2 = \hat{n}$, which is the normal to the interface, such that:

$$\boxed{\vec{D}_{2n} - \vec{D}_{1n} = \sigma \qquad . . . (1)}$$

Discontinuity for the normal component of the flux

تشير المعادلة الاخيرة الا ان اللااستمرارية في المركبة العمودية لكثاقة الفيض على الحد الفاصل بين وسطين تعتمد على كثافة الشحنة السطحية للشحنات الحرة للسطح، وان لم تتواجد شحنات حرة فان هذه المركبه ستكون مستمرة عند ذلك السطح.

Now, we have the field – potential correlation: $\vec{E} = -\vec{\nabla}U$, which means that in one dimension form, $\vec{E} = -\frac{d}{dl}U\hat{\imath} \rightarrow \vec{E}.d\vec{l} = -dU\hat{\imath} = 0$, i.e. the path around any closed loop into the interface vanishes. Applying this fact in the following fig. yields;



$$\int_A^B \vec{E}_2.\Delta\vec{l}_{AB} + \int_B^C \vec{E}.\Delta\vec{l}_{BC} + \int_C^D \vec{E}_1.\Delta\vec{l}_{CD} + \int_D^A \vec{E}.\Delta\vec{l}_{DA} = 0$$

According our earliear assmption that pillbox high is small, $\left.\begin{matrix}\Delta l_{BC}\\ \Delta l_{DA}\end{matrix}\right\} \rightarrow 0$ in the 2nd and 4th terms, thus;

$$\int_A^B \vec{E}_2.\Delta\vec{l}_{AB} + \int_D^C \vec{E}_1.(-\Delta\vec{l}_{AB}) = 0$$

$$\int_A^B (\vec{E}_2 - \vec{E}_1).\Delta\vec{l}_{AB} = 0$$

$$(\vec{E}_2 - \vec{E}_1).\Delta\vec{l}_{AB} = 0$$

$$\boxed{\rightarrow E_{2t} = E_{1t} \quad ... \quad (2)}$$

Continuity for the tangential component of electric field

According to eq.(2) we can also write the equation of *potential continuity*;

$$\boxed{U_{2t} = U_{1t} \quad ... \quad (3)}$$

Note:

Equations (1), (2) & (3) have been obtained for two arbitrary media. If medium 1 is taken as a conductor, then $\vec{E}_1 = 0$. i.e. there is no polarization, $\vec{P}$, and the electric displacement vanishing;

$$\vec{D} = \epsilon_o \vec{E}_1 + \vec{P} = 0$$

thus eq.(1) and (2) will be;

$$\boxed{\begin{array}{ll} \vec{D}_{2n} = \sigma & \quad ...\ (4) \\ E_{2t} = 0 & \quad ...\ (5) \end{array}}$$

Displacement and electric field
in a dielectric just outside of a conductor

Example: Schaum p.95

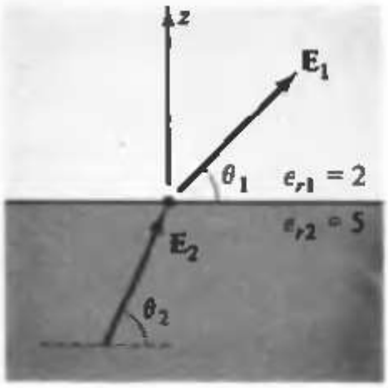
Given that $\vec{E}_1 = 2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}\ V/m$, at the charge free dielectric interface of the following fig. Find $\vec{D}_2$ and the angles θ_1 and θ_2.

For two charge free dielectric media located in contact and having $\varepsilon_{r1} = 2$, and $\varepsilon_{r2} = 5$, given that $\vec{E}_1 = 2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}\ V/m$, inside medium 1 and directed away from the interface, and making an angle of θ_1 with the horizontal axis. Find $\vec{D}_2$ for the incoming field passing within medium 2 and the angle θ_2 for it with the parallel to the horizontal interface.

Solution:

ابتداءا يجب تحديد مركبات المجال الثلاث كعمودية ومماسيه اعتمادا على موقع الحد الفاصل interface ، من الرسم يتضح بأن مركبة zهي مركبة عمودية، ومن معادلة سابقة نحن نعلم بأن المركبة العمودية هي غير مستمرة. اما المركبتان المتبقيتان فأنهما مماسيتان، اي انهما وفقا لمعادلة سابقة، يكونان مستمرتان للوسطين.

the interface is a z −constant plan. The x and y components are both tangential components, and the z-component are normal. The medium is charge-free, thus $\sigma = 0$, i.e. $\vec{D}_{2n} - \vec{D}_{1n} = 0$ which means that the normal component becomes continues



continuity of the tangential component of $\vec{E}$ and the normal component of $\vec{D}$ gives:

$$\vec{E}_1 = 2\hat{\imath} - 3\hat{\jmath} + 5\hat{k} \quad . . . (1)$$

$$\vec{E}_2 = 2\hat{\imath} - 3\hat{\jmath} + E_{2z}\hat{k} \quad . . . (2)$$

but: $\vec{D} = \varepsilon_o\varepsilon_r\vec{E}$, substituting in eq.(1) & (2) yields;

$$\vec{D}_1 = (\varepsilon_o\varepsilon_{r1})2\hat{\imath} - (\varepsilon_o\varepsilon_{r1})3\hat{\jmath} + (\varepsilon_o\varepsilon_{r1})5\hat{k}$$

$$= \varepsilon_o(2)2\hat{\imath} - \varepsilon_o(2)3\hat{\jmath} + \varepsilon_o(2)5\hat{k}$$

$$\rightarrow \vec{D}_1 = \varepsilon_o(4)\hat{\imath} - \varepsilon_o(6)\hat{\jmath} + \varepsilon_o(10)\hat{k} \quad . . . \quad (3)$$

also;

$$\vec{D}_2 = (\varepsilon_o\varepsilon_{r2})2\hat{\imath} - (\varepsilon_o\varepsilon_{r2})3\hat{\jmath} + (\varepsilon_o\varepsilon_{r2})E_{2z}\hat{k}$$

$$\rightarrow \vec{D}_2 = \varepsilon_o(5)2\hat{\imath} - \varepsilon_o(5)3\hat{\jmath} + \varepsilon_o(5)E_{2z}\hat{k} \quad . . . \quad (4)$$

We have: $\vec{D}_{2n} - \vec{D}_{1n} = \sigma$, but the interface is charge –free, $\sigma = 0$, thus;

$$\vec{D}_{z2n} = \vec{D}_{z1n}$$

Comparing the field z component in eqs.(3) & (4), we get;

$$\varepsilon_o(10) = \varepsilon_o(5)E_{2z}$$

$$\rightarrow \quad E_{2z} = 2 \quad . . . \quad (5)$$

The angelsθ_1 and θ_2 between the interface and the two fields $\vec{E}_1$ and $\vec{E}_2$ are given as;

$$\vec{E}_1 . \hat{k} = |\vec{E}_1|\cos(90 - \theta_1)$$

From eq.(1); and the identity: cos (A - B) = (cos A)(cos B) + (sin A)(sin B)

$$\vec{E}_1.\hat{k} = 5 = \sqrt{(2)^2 + (-3)^2 + (5)^2}[sin\theta_1]$$

$$\rightarrow\ sin\theta_1 = \frac{5}{\sqrt{38}}$$

$$\therefore\ \ \theta_1 = 54.2^o$$

also;

$$\vec{E}_2.\hat{k} = |\vec{E}_1|\cos(90 - \theta_2)$$

From eq.(2) (including eq. (5)), and the same identity;

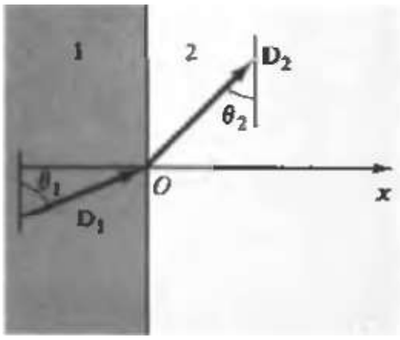
$$2 = \sqrt{(2)^2 + (-3)^2 + (2)^2}[sin\theta_1]$$

$$\rightarrow\ sin\theta_2 = \frac{2}{\sqrt{17}}$$

$$\therefore\ \ \theta_2 = 29.0^o$$

H.W.

Region 1, defined by $x < 0$, is free space, while region 2, $x > 0$, is a dielectric material for which $\varepsilon_{r2} = 2.4$.



4 – 7: Boundary - value problems involving dielectrics:

If the dielectric with which we are concerned is a linear, isotropic and homogeneous, then; $\vec{D} = \epsilon\vec{E}$, where ϵ is const. charactristics the material, substituting in the differential form of Gauss,s law in a dielectric;

$$\vec{\nabla}.\vec{D} = \rho_{free}$$

where ρ is free charge density.

$$\vec{\nabla}.\epsilon\vec{E} = \rho_{free}$$

$$\vec{\nabla}.\vec{E} = \frac{1}{\epsilon}\rho_{free}$$

also; $\vec{E} = -\vec{\nabla} U$, then;

$$\vec{\nabla}.\left(-\vec{\nabla} U\right) = \frac{1}{\epsilon}\rho_{free}$$

$$\boxed{\nabla^2 U = -\frac{1}{\epsilon}\rho_{free} \quad . . . (6)}$$

The potential equation in dielectrics, (Poisson's equation)

Thus, the potential satisfies Poisson's equation.

Cases:

1. In *vacuum*, ϵ replaces by ϵ_o.
2. ***Inside*** **dielectric**, no free charges found, i.e. $\rho = 0$, thus, the potential equation satisfies *Laplace's* one, $\nabla^2 U = 0$.
3. ***Inside*** **conductor**, no free charges found, $\rho = 0$, instead, it contains point charges, $\rho \to \sum_i q_i$, and surface charges. The potential equation transforms to be *Laplace's* one

Figure 4.20

Example 4.5

A metal sphere of radius a carries a charge Q (Fig. 4.20). It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).

Solution: To compute V, we need to know **E**; to find **E**, we might first try to locate the bound charge; we could get the bound charge from **P**, but we can't calculate **P** unless we already know **E** (Eq. 4.30). We seem to be in a bind. What we *do* know is the *free* charge Q, and fortunately the arrangement is spherically symmetric, so let's begin by calculating **D**, using Eq. 4.23:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \qquad \text{for all points } r > a.$$

(Inside the metal sphere, of course, $\mathbf{E} = \mathbf{P} = \mathbf{D} = 0$.) Once we know **D**, it is a trivial matter to obtain **E**, using Eq. 4.32:

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } a < r < b, \\[2ex] \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > b. \end{cases}$$

The potential at the center is therefore

$$\begin{aligned} V &= -\int_{\infty}^{0} \mathbf{E}\cdot d\mathbf{l} = -\int_{\infty}^{b}\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)dr - \int_{b}^{a}\left(\frac{Q}{4\pi\epsilon r^2}\right)dr - \int_{a}^{0}(0)\,dr \\ &= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right). \end{aligned}$$

As it turns out, it was not necessary for us to compute the polarization or the bound charge explicitly, though this can easily be done:

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \frac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}.$$

in the dielectric, and hence

$$\rho_b = -\nabla \cdot \mathbf{P} = 0,$$

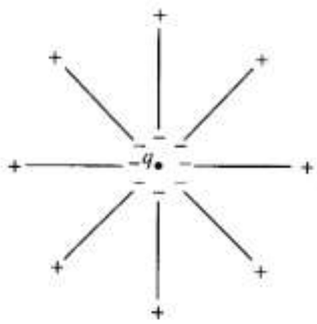
while

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0 \chi_e Q}{4\pi\epsilon b^2}, & \text{at the outer surface,} \\ \dfrac{-\epsilon_0 \chi_e Q}{4\pi\epsilon a^2}, & \text{at the inner surface.} \end{cases}$$

Notice that the surface bound charge at a is *negative* ($\hat{\mathbf{n}}$ points outward *with respect to the dielectric,* which is $+\hat{\mathbf{r}}$ at b but $-\hat{\mathbf{r}}$ at a). This is natural, since the charge on the metal sphere attracts its opposite in all the dielectric molecules. It is this layer of negative charge that reduces the field, within the dielectric, from $1/4\pi\epsilon_0(Q/r^2)\hat{\mathbf{r}}$ to $1/4\pi\epsilon(Q/r^2)\hat{\mathbf{r}}$. In this respect a dielectric is rather like an imperfect conductor: on a *conducting* shell the induced surface charge would be such as to cancel the field of Q *completely* in the region $a < r < b$: the dielectric does the best it can, but the cancellation is only partial.

4 – 8: Dielectric sphere in a uniform electric field:

4 – 9: Force in a point charge embedded in a dielectric:



Solved problems: (Schaum ch.7)

1. Find the magnitudes of $\vec{P}$ and $\vec{D}$ for a dielectric material in which $E = 0.15 Mev/m$, and $\chi_e = 4.25$.

We have:

$$\vec{P} = \chi\vec{E} = \chi_e \varepsilon_o \vec{E}$$

where: $\chi_e = \frac{\chi}{\varepsilon_o}$. Sub. in eq. of $\vec{D}$, yields; $\vec{D} = \varepsilon_o \varepsilon_r \vec{E}$

$$\rightarrow \varepsilon_r = 1 + \chi_e$$
$$= 5.24$$

$\vec{D} = \varepsilon_o \varepsilon_r \vec{E}$

$$= 6.96 \mu C/m^2$$

$\vec{P} = \chi_e \varepsilon_o \vec{E}$

$$= 5.64 \mu C/m^2$$

Solved Problems

7.1. Find the polarization **P** in a dielectric material wtih $\epsilon_r = 2.8$ if $\mathbf{D} = 3.0 \times 10^{-7}\mathbf{a}\ \mathrm{C/m^2}$.

Assuming the material to be homogeneous and isotropic,

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}$$

Since $\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$ and $\chi_e = \epsilon_r - 1$,

$$\mathbf{P} = \left(\frac{\epsilon_r - 1}{\epsilon_r}\right) D = 1.93 \times 10^{-7}\mathbf{a}\ \mathrm{C/m^2}$$

7.2. Determine the value of **E** in a material for which the electric susceptibility is 3.5 and $\mathbf{P} = 2.3 \times 10^{-7}\mathbf{a}\ \mathrm{C/m^2}$.

Assuming that **P** and **E** are in the same direction,

$$\mathbf{E} = \frac{1}{\chi_e \epsilon_0}\mathbf{P} = 7.42 \times 10^{3}\mathbf{a}\ \mathrm{V/m}$$

7.3. Two point charges in a dielectric medium where $\epsilon_r = 5.2$ interact with a force of 8.6×10^{-3} N. What force could be expected if the charges were in free space?

Coulomb's law, $F = Q_1 Q_2/(4\pi\epsilon_0\epsilon_r d^2)$, shows that the force is inversely proportional to ϵ_r. In free space the force will have its maximum value

$$F_{max} = \frac{5.2}{1}(8.6 \times 10^{-3}) = 4.47 \times 10^{-2}\ \mathrm{N}$$

7.4. Region 1, defined by $x < 0$, is free space, while region 2, $x > 0$, is a dielectric material for which $\epsilon_{r2} = 2.4$. See Fig. 7-9. Given

$$\mathbf{D}_1 = 3\mathbf{a}_x - 4\mathbf{a}_y + 6\mathbf{a}_z \quad \mathrm{C/m^2}$$

find $\mathbf{E}_2$ and the angles θ_1 and θ_2.

The x components are normal to the interface: D_n and E_t are continuous.

$$\mathbf{D}_1 = 3\mathbf{a}_x - 4\mathbf{a}_y + 6\mathbf{a}_z \qquad \mathbf{E}_1 = \frac{3}{\epsilon_0}\mathbf{a}_x - \frac{4}{\epsilon_0}\mathbf{a}_y + \frac{6}{\epsilon_0}\mathbf{a}_z$$

$$\mathbf{D}_2 = 3\mathbf{a}_x + D_{y2}\mathbf{a}_y + D_{z2}\mathbf{a}_z \qquad \mathbf{E}_2 = E_{x2}\mathbf{a}_x - \frac{4}{\epsilon_0}\mathbf{a}_y + \frac{6}{\epsilon_0}\mathbf{a}_z$$