Chapter Seven

The Magnetic Field of Steady Currents

The second kind of field which enters into the study of electricity and magnetism is, of course, the magnetic field. Such fields or, more properly, the effects of such fields have been known since ancient times when the effects of the naturally occurring permanent magnet magnetite (Fe_3O_4) were first observed. The discovery of the north- and south-seeking properties of this material had a profound influence on early navigation and exploration. Except for this application, however, magnetism was a little used and still less understood phenomenon until the early nineteenth century, when Oersted discovered that *an electric current produced a magnetic field*. This work, together, with the later work of Gauss, Henry, Faraday and others, has brought the magnetic field into prominence as a partner to the electric field.

In this chapter the basic definitions of magnetism will be given, the production of magnetic fields by *steady currents* will be studied .and some important groundwork for future work will be laid.

7-1 The definition of Magnetic Induction

For the purpose of defining the magnetic induction (magnetic flux density) (B) it is convenient to define the *magnetic force,* $\vec{F}_m$, (frequently called the *Lorentz force*), as *that part of the force exerted on a moving charge in a steady magnetic field which is neither electrostatic nor mechanical*, where that force is perpendicular to each B, and charge velocity ($\vec{v}$).

The B, is then defined as *the vector which satisfies*;

$$\vec{F}_m = \lim_{q \to 0} q\vec{v} \times \vec{B} \quad . \quad . \quad . \quad (*)$$

وجود اداة الضرب الاتجاهي في العلاقة السابقة يحتم ان تكون النتيجة عمودية على كل من سرعة الشحنة واتجاه المجال المغناطيسي، وهذا يتطابق مع قاعدة اليد اليمنى.

where the limit used to ensure the q <u>*does not affect the source of B*</u>.

For simplicity we can write;

$$\vec{F}_m = q\vec{v} \times \vec{\mathrm{B}} \quad . \quad . \quad . \quad (7-1)$$

The unit for magnetic induction in the "mks" system is *Tesla (T)*, where according to Eq. (6-1):

$$1Tesla = 1\frac{N.Sec}{C.m} = 1\frac{N}{A.m}$$

It is customary to express this unit as the Weber/meter ^{2}; the Weber is the mks unit of *magnetic flux* which will be defined later.

<u>H.W:</u>

Prove that:

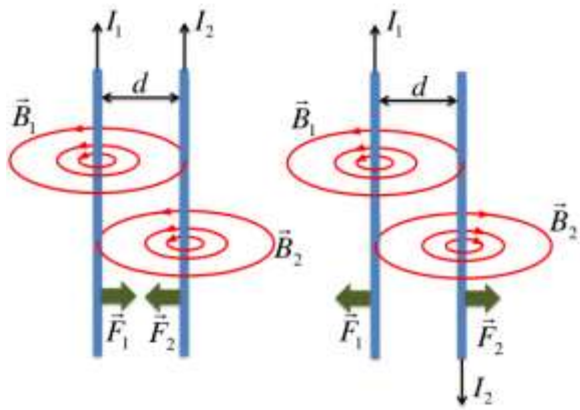
$$\frac{N.Sec}{C.m} = \frac{Newton}{A.m} = \frac{Weber}{meter^2}$$

And often the magnetic field is given in Gauss (G), the CGS unit.

Consider two parallel straight wires in which two *steady currents* are flowing;

- If the wires are neutral, there is no net *electric force* between the two wires.
- If the current in both wires is flowing in the same direction, the wires are found to *attract* each other.
- If the current in one of those wires is reversed, the wires are found to *repel* each other.

The force responsible for the attraction and repulsion is called the **magnetic force.** The magnetic force acting on a moving charge q is defined in terms of the **magnetic field**.



ان سريان تيار كهربائي مستمر steady في سلك يؤدي الى توليد مجال مغناطيسي مستمر steady حول ذلك السلك، واعتمادا على اتجاه ذلك التيار يتحدد اتجاه المجال المغناطيسي الناشيء (تجربة اورستد: اتجاه التيار يتمثل بالابهام، اتجاه المجال المغناطيسي المطلوب يتمثل بلفه الاصابع الاربعة). فاذا كان اتجاه التيار للسلكين متماثلان، فالمجال المغناطيسي المتولد سيكون بنفس الاتجاه والقوة بين السلكين تصبح قوة تجاذب. وبنفس الطريقه تكون القوة بين السلكين تنافرية عند سريان التيار باتجاهين متعاكسان.

<u>Result:</u>

For a wire carrying steady current, two fields will exists with two different plans:

- The electric field *<u>diverges</u>* from the line charge (current carriers inside the wire) and is *curl free*: $(\nabla \times \vec{E} = 0)$.

- The magnetic field *<u>forms circles</u>* around the steady current and is divergence free: $(\nabla . \vec{B} = 0)$.

7-2 Magnetic Force and Torque on Current-Carrying Conductor:

Perfectly good definitions of the magnetic induction can be constructed by using the force on a current element or the torque on a current-carrying loop. So, from the definition of B, an expression for the force on an element $d\ell$ of a current-carrying conductor can be found.

If $d\ell$ is an element of conductor with its sense taken in the direction of the current I which it carries, then $d\ell$ is parallel to the velocity v of the charge carriers in the conductor. If there are N charge carriers per unit volume in the conductor, the force on the element $d\ell$ is;

$$d\vec{F}_m = NA|d\ell|q\vec{v} \times \vec{B} \quad . \; . \; . \quad (7-2)$$

Where A is the cross-sectional area of the conductor and q is the charge per carrier. Since $\vec{v}$ and $d\ell$ are parallel, an alternative form of equation (7-2) can written as follows;

$$d\vec{F}_m = Nq|\vec{v}\,|A\; dl \times \vec{B} \quad . \; . \; . \quad (7-3)$$

However, $(Nq|\vec{v}\,|A)$ is just the current I for a single species of carrier. Therefore the expression:

$$\boxed{d\vec{F}_m = I\; dl \times \vec{B} \quad . \; . \; . \quad (7-4)}$$

is written for the force on an infinitesimal element of a charge-carrying conductor.

Equation (7-4) can be integrated to give the force on a complete (or closed) circuit. If the circuit in question, represented by the contour C, then;

$$\vec{F}_m = \oint_C I\; dl \times \vec{B} \quad . \; . \; . \quad (7-5)$$

Assuming B is uniform (not depend on position) then both B and I can removed from eq. under integral and then equation (7-5) becomes;

$$\vec{F}_m = I\left\{\oint_C \; dl\right\} \times \vec{B} \quad . \; . \; . \quad (7-6)$$

The remaining integral is easy to evaluate. Since it is the sum of infinitesimal vectors forming a complete circuit, it must be zero. Thus;

$$\vec{F}_m = I\left\{\oint_C \; dl\right\} \times \vec{B} \; = 0 \quad \therefore \vec{B} \; is \; uniform \quad . \; . \; . \quad (7-7)$$

Another interesting quantity is the torque on a complete circuit. Since torque is moment of force, the infinitesimal torque $d\tau$ is given by;

$$\boxed{d\vec{\tau} = \vec{r} \times d\vec{F}_m \quad . \; . \; . \quad (7-8)}$$

$$= \vec{r} \times I\left(d\vec{\ell} \times \vec{B}\right)$$

The torque on a complete circuit is;

$$\vec{\tau} = I\oint_C \; \vec{r} \times \left(d\vec{\ell} \times \vec{B}\right) \quad . \; . \; . \quad (7-9)$$

The operation between the brackets could be fined by a matrix of cross product:

$$d\vec{\ell} \times \vec{B} = \hat{\imath}\left(dyB_z - dzB_y\right) + \hat{\jmath}(dzB_x - dxB_z) + \hat{k}\left(dxB_y - dyB_x\right) \; . \; . (7-10)$$

The same procedure, we can find the result of: $\vec{r} \times \left(d\vec{\ell} \times \vec{B}\right)$;

$$\left.\begin{aligned} \left\{\vec{r} \times \left(d\vec{\ell} \times \vec{B}\right)\right\}_x &= \; ydxB_y - \; ydyB_x - \; zdzB_x + zdxB_z \\ \left\{\vec{r} \times \left(d\vec{\ell} \times \vec{B}\right)\right\}_y &= \; zdyB_z - \; zdzB_y - \; xdxB_y + xdyB_x \\ \left\{\vec{r} \times \left(d\vec{\ell} \times \vec{B}\right)\right\}_z &= \; xdzB_x - \; xdxB_z - \; ydyB_z + ydzB_y \end{aligned}\right\} \; . \; . \; . \; (7-11)$$

B is assumed to be independent of r (uniform field), the x-component of the torque being;

$$\tau_x = I\oint_C \; \left\{\vec{r} \times \left(d\vec{\ell} \times \vec{B}\right)\right\}_x \quad . \; . \; . \quad (7-12)$$

Using equation (7-11);

$$\tau_x = I \oint_C \{ydxB_y - ydyB_x - zdzB_x + zdxB_z\} \quad . . . \quad (7-13)$$

$$= I\left\{B_y \oint_C ydx - B_x \oint_C ydy - B_x \oint_C zdz + B_z \oint_C zdx\right\} \quad . . . (7-14)$$

$$= I\left\{B_y \oint_C ydx + B_z \oint_C zdx\right\} \quad . . . \quad (7-15)$$

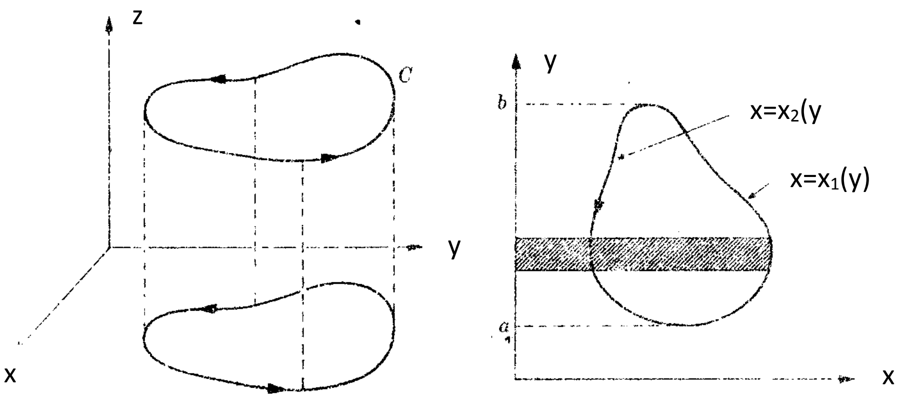


Figure 1:

$$\oint_C ydx = \int_a^b ydx + \int_b^a y_2 dx = A_z \quad . . . \quad (7-16)$$

Accordingly equation (7-15) becomes;

$$\tau_x = I\{A_y B_z - A_z B_y\} \quad . . . \quad (7-17a)$$

Similarly, for the two remaining components, we can find that;

$$\tau_y = I\{A_z\, B_x - A_x B_z\} \quad . . . \quad (7-17b)$$

$$\tau_z = I\{\mathrm{A_z B_y - A_y B_x}\} \quad . . . \quad (7-17c)$$

Thus;

$$\vec{\tau} = I\, \vec{\mathrm{A}} \times \vec{\mathrm{B}} \quad . . . \quad (7-18)$$

Magnetic torque

Where A is the vector whose components are the areas enclosed by projections of the curve C on the yz-, zx-, and xy-planes. The quantity IA appears very frequently in magnetic theory, and is referred to the *magnetic moment* of the circuit. The symbol m will be used for magnetic moment:

$$m = I\,\vec{A} \quad . \;\; . \;\; . \quad (7-19)$$

Magnetic moment

It is easy to show, by the technique used above, that the integral of $(\vec{r} \times d\vec{\ell})$ around a closed (electric circuit) path gives twice the area enclosed by the curve. Thus;

$$\frac{1}{2} I \oint_C \vec{r} \times d\vec{\ell} = \vec{A} \quad . \;\; . \;\; . \quad (7-20) \quad \boldsymbol{H.W}$$

If the current exist inside a medium, $Id\vec{\ell} \rightarrow \vec{J}dv$, also;

$$m = \frac{1}{2} I \oint_C \vec{r} \times d\vec{\ell} \quad . \;\; . \;\; . \quad (7-21)$$

7-3 Biot-Savart Law

على غرار صيغه قانون كولومب (في الكهربائية الساكنة) لحساب *المجال الكهربائي الساكن* *steady* المتولد بسبب شحنة كهربائية ثابته (او متحركة بسرعة ثابته) $\vec{E} = k\frac{q}{r^2}\hat{r}$.

في المغناطيسية الساكنة اوجد بايوت وسافارت صيغه رياضية لحساب *المجال المغناطيسي الساكن* *steady* المتولد بسبب شحنة كهربائية متحركة بسرعه ثابته (كحركة الكترون في مدار ارضي حول الذرة)، والصيغه الرياضية لهذا القانون تتناسب مع الشحنة (او التيار) وسرعتها وموقعها.

The Biot-Savart law defines the magnetic field $\vec{B}$ due a point charge q moving with a velocity $\vec{v}$ as;

$$\vec{B} = \frac{\mu_o}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^3} = \frac{\mu_o}{4\pi}\frac{q\vec{v}\times\vec{r}}{|\vec{r}|^3}$$

Here, $\hat{r}$ is a unit vector that points from the position of the charge to the point at which the field is evaluated, r is the distance between the charge and the point at which the field is evaluated, and the number $\mu_o/4\pi$ $(= 10^{-7} N/A^2)$ which appears in last equation plays the same role here as $1/4\pi\varepsilon_o$ played in electrostatic, i.e. it is the constant which is required to make an experimental law compatible with a set of units.

For a steady electric current moving in a wire within a circuit, the law takes the following two general forms:

$$\vec{B} = \frac{\mu_o}{4\pi} I \oint \frac{d\vec{\ell}\times\vec{r}}{|\vec{r}|^3}$$

Integral form

Or with reference to the origin:

$$\vec{B}(r) = \frac{\mu_o}{4\pi} I \oint \frac{d\vec{\ell}\times(\vec{r}_2-\vec{r}_1)}{|(\vec{r}_2-\vec{r}_1)|^3}$$

Last equation can be modified to express the magnetic field $(d\vec{B})$ due to an infinitesimal current element $\vec{\ell}$, which can be written as,

$$d\vec{B} = \frac{\mu_o}{4\pi}\frac{Id\vec{\ell}\times\vec{r}}{|\vec{r}|^2}$$

Differential form

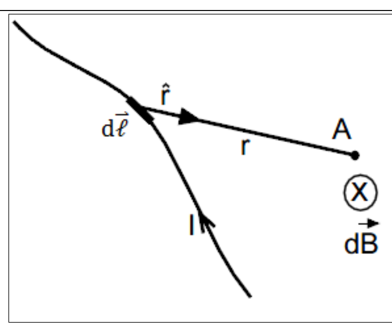


Illustration of Biot-Savart law, electric current form.

The quantities in last eq. are illustrated in figure. The direction of the magnetic field due to the current element at the point A can be inferred using the right hand rule. $d\vec{B}$ is directed into the plane of the figure at A. The field due to the entire wire can be evaluated at A by adding up the contributions of all the current elements in the wire.

7-4 Elementary applications of the Biot and Savart law

1) **Long straight wire:**

Assume the wire located along the *x*-axis, extended from minus infinity to plus infinity and carry a current *I*. The magnetic field will be computed at a typical point r_2 on the *y*-axis. The geometry is best explained in figure bellow;

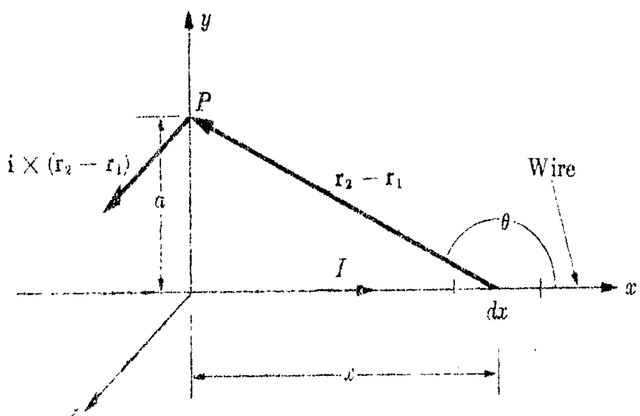


Figure 6-3: Schaum p.136 Magnetic field at point P due to a long straight wire.

The magnetic induction is just;

$$\vec{B}(r_2) = \frac{\mu_o}{4\pi} I \int_{-\infty}^{+\infty} \frac{dx\ \hat{\imath} \times (\vec{r}_2 - \vec{r}_1)}{|(\vec{r}_2 - \vec{r}_1)|^3} \quad . \ . \ . \quad (7-29)$$

First of all we need to solve the cross product, so, $(\vec{r}_2 - \vec{r}_1)$ is ling in the xy-plane of the problem, then the result of its product should be normal $(\hat{k})$to this plan:

$$\therefore\ \hat{\imath} \times (\vec{r}_2 - \vec{r}_1) = |\hat{\imath}||(\vec{r}_2 - \vec{r}_1)|\ sin\theta\ \hat{k} \qquad 0 \le \theta \le \pi \qquad (*)$$

In order to normalize vectors and angels of the problem, we will change vectors to angels:

$$tan(\pi - \theta) = -tan\theta = \frac{a}{x} \qquad (**)$$

And,

$$\csc(\pi - \theta) = \csc(\theta) = \frac{|(\vec{r}_2 - \vec{r}_1)|}{a} \qquad (***)$$

Substituting eqs. *,**, ***, in eq.(7-29), yields: **(H.W)**

$$\vec{B}(r_2) = \frac{\mu_o}{4\pi} I\hat{k}\frac{1}{a}\int_0^{\pi} sin\theta\ d\theta = \frac{\mu_o I}{4\pi a}\hat{k}(-cos\theta)\Big|_0^{\pi}$$

$$\boxed{\therefore\ \vec{B}(r_2) = \frac{\mu_o I}{2\pi a}\hat{k} \quad . . . \ (7-30)}$$

To use this result more generally, it is only necessary to note that the problem exhibits an obvious symmetry about the x-axis. Thus we conclude that the *lines of B are everywhere circles*; with the conductor as a center. This is in complete agreement with the elementary result which gives the direction of B by a right-hand rule.

2) Circular wire: p.218 Griffths

The magnetic field produced by such a circuit at an arbitrary point is very difficult to compute. However, if only points on the axis of symmetry are considered, the expression for B is relatively simple. In this example a complete vector treatment will be used to demonstrate the technique. Figure 7-4 illustrates the geometry and the coordinates to be used.

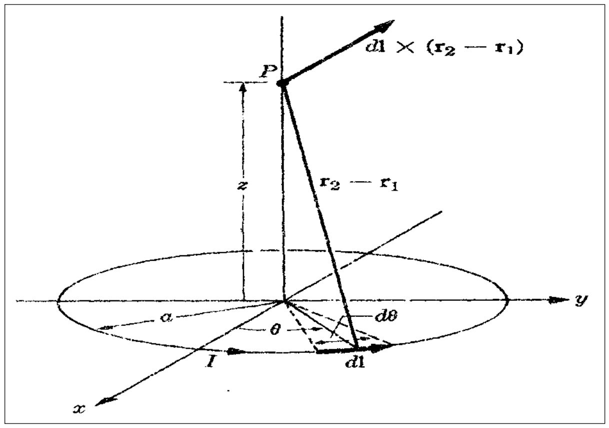


Figure 7-4: Magnetic field at point P due to a circular turn of wire.

The field is to be calculated at point r_2 on the z-axis; the circular turn lies in the xy-plane. The magnetic induction is given by equation (6-23) in which, from figure 6-4, the following expressions are to be used:

$$d\vec{\ell} = a\, d\theta(-\hat{\imath}\, sin\theta + \hat{\jmath}\, cos\theta),$$

$$(\vec{r}_2 - \vec{r}_1) = -\hat{\imath}\, a\, cos\theta - \hat{\jmath}\, a\, sin\theta + \hat{k}\, z$$

And Pythagorean theorem:

$$|(\vec{r}_2 - \vec{r}_1)| = (a^2 + z^2)^{1/2}$$

Substituting these equations into equation (6-23), yields:

$$\vec{B}(z) = \frac{\mu_o I}{4\pi} \int_o^{2\pi} \frac{\left(\hat{\imath}\, z\, a\, cos\theta + \hat{\jmath}\, z\, asin\theta + \hat{k}a^2\right)}{(z^2 + a^2)^{\frac{3}{2}}} d\theta \quad . \; . \; . \qquad (7-31)$$

The 1st two terms integrate to zero,

$$\boxed{\vec{B}(z) = \frac{\mu_o I}{2} \frac{a^2}{(z^2 + a^2)^{\frac{3}{2}}} \hat{k} \quad . \; . \; . \quad (6-32)}$$

Field along the z-axis.

7-4: Magnetic force between two circuits (Ampere Force Law):

In 1820, just a few weeks after Oersted announced his discovery that currents produce magnetic effects, Ampere presented the results of series of experiments which may be generalized and expressed in modern mathematical language as;

$$\vec{F}_2 = \frac{\mu_o}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\vec{\ell}_2 \times [d\vec{\ell}_1 \times (\vec{r}_2 - \vec{r}_1)]}{|\vec{r}_2 - \vec{r}_1|^3} \quad . . . \quad (7-22)$$

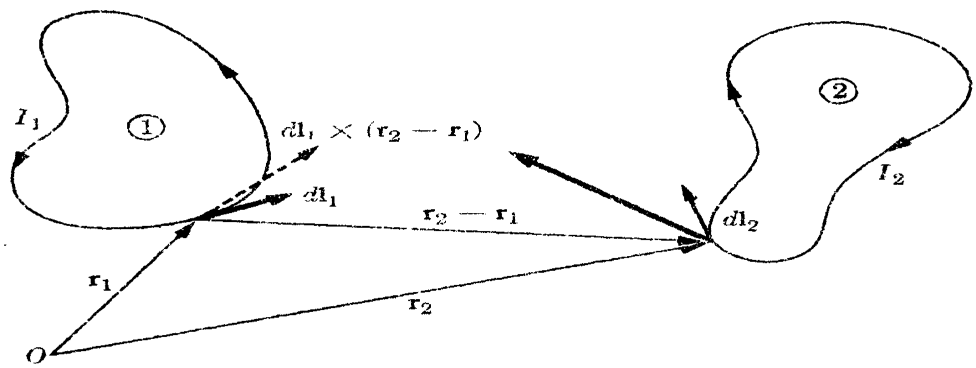


Figure 2: Magnetic force between two circuits.

التيارات في الدائرتين تتجهان بنفس الاتجاه عليه وطبقا لقاعدة اليد اليمنى فان القوة المغناطيسية بينهما هي قوة تجاذب وكما يتضح ذلك من الشكل

This rather formidable expression can be understood with reference to figure 2. The force F_2 is the force exerted on circuit 2 due to the influence of circuit 1, the $d\ell$'s and r's are explained by the figure.

<u>H.W:</u> Show that (Newton's 3ed law);

$$\vec{F}_1 = -\vec{F}_2$$

Eqs. (7-5) & (7-22), for the last two electric circuits, Biot-Savart two forms could be re-writing as the special forms:

$$\vec{B}(r_2) = \frac{\mu_o}{4\pi} I_1 \oint_1 \frac{d\vec{\ell}_1 \times (\vec{r}_2 - \vec{r}_1)}{|(\vec{r}_2 - \vec{r}_1)|^3} \quad . . . \quad (7-23)$$

$$d\vec{B}(r_2) = \frac{\mu_o}{4\pi} \frac{I_1 d\vec{\ell}_1 \times (\vec{r}_2 - \vec{r}_1)}{|(\vec{r}_2 - \vec{r}_1)|^3} \quad . . . \quad (7-24)$$

Integral and Differential forms of Biot-Savart law

The magnetic field exerted on circuit 2 due to the influence of circuit 1

For continuous distribution of current, last two equations can be written in term of current density J, as;

$$\vec{B}(r_2) = \frac{\mu_o}{4\pi}\oint_1 \frac{\vec{J}(r_2) \times (\vec{r}_2 - \vec{r}_1)}{|(\vec{r}_2 - \vec{r}_1)|^3}\, dv_1 \quad . \; . \; . \quad (7-25)$$

$$d\vec{B}(r_2) = \frac{\mu_o}{4\pi}\frac{\vec{J}(r_2) \times (\vec{r}_2 - \vec{r}_1)}{|(\vec{r}_2 - \vec{r}_1)|^3} dv_1 \quad . \; . \; . \quad (7-26)$$

From eq. (2-25), It is possible mathematically, verify that:

$$\nabla . \vec{B} = 0 \quad . \; . \; . \quad (7-*)$$

An isolated magnetic pole cans never being existed

The Proof:

By using the identity, with eq. (7-25);

$$div(\vec{A} \times \vec{B}) = -\vec{A} \cdot curl\, \vec{B} + \vec{B} \cdot curl\, \vec{A}$$

Yields;

$$div_2\, \vec{B}(r_2) =$$

$$\frac{\mu_o}{4\pi}\oint_1 \left[-\vec{J}(r_2) \cdot \mathrm{curl}_2 \frac{(\vec{r}_2 - \vec{r}_1)}{|(\vec{r}_2 - \vec{r}_1)|^3} + \frac{(\vec{r}_2 - \vec{r}_1)}{|(\vec{r}_2 - \vec{r}_1)|^3} \cdot \mathrm{curl}_2\big(\vec{J}(r_2)\big)\right] dv$$

$$. \; . \; . \quad (7-27)$$

But the *electric current is non-rotational*, so the second term vanishes, furthermore; $\vec{\nabla}\frac{1}{|(\vec{r}_2-\vec{r}_1)|} = -\frac{(\vec{r}_2-\vec{r}_1)}{|(\vec{r}_2-\vec{r}_1)|^3}$, and *the curl of any gradient is zero*, it follows that;

$$div_2\, \vec{B}(r_2) = 0 \quad . \; . \; . \quad (7-28)$$

7-5: Ampere's Circuital Law:

Similar to Gauss's law, Ampere's law states that the line integral of the tangential components of *magnetic field strength* (H) around a closed path is the same as the net current enclosed by the path.

The curl of magnetic induction fields given by equations (7-23) or (7-25) which are due to steady currents, i.e. to currents which satisfy **div J** = 0, defiantly lead to the equation;

$$curl\,\vec{B}(r_2) = \mu_o \vec{J}(r_2) \quad . . . (7-33)$$

Differential form of Ampere's law

This equation also called ***Ampere's Circuital Law****.*

In fact this equation is still valid as long as there are no magnetic material present and **div J** = 0. However, there is another form for this law called integral form of Ampere's law which may derive as follows;

the integration of equation (7-33) over the total surface bounded by the circuit leads to;

$$\int_S \; curl\,\vec{B} \cdot \hat{n}da = \mu_o \int_S \; \vec{J} \cdot \hat{n}da \; . . . (7-34)$$

Using the Stokes' theorem $\int_S \; curl\,\vec{B} \cdot \hat{n}da = \oint_C \; \vec{B} \cdot d\vec{\ell}$ for the right hand side of the last equation we get:

$$\oint_C \; \vec{B} \cdot d\vec{\ell} = \mu_o \int_S \; \vec{J} \cdot \hat{n}da$$

$$\oint_C \; \vec{B} \cdot d\vec{\ell} = \mu_o I \quad . . . (7-36)$$

The integral form for Ampere's law

which state that *{the line integral of* ***B*** *around a closed path is equal to* μ_o *times the total current through the closed path}*.

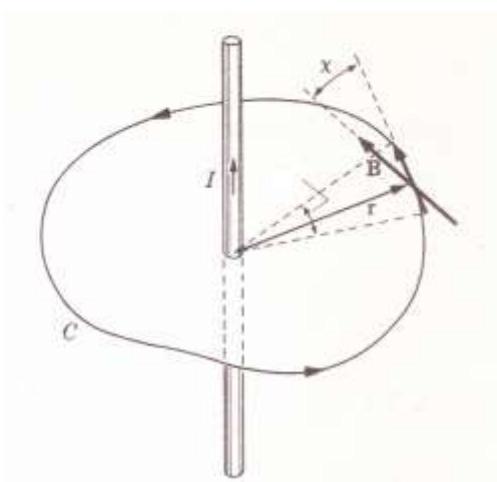
It is clear that Ampere's circuital law, as equation (7-36) is called, is *parallel to Gauss's law in electrostatics*. By this is meant that it can be used to obtain the magnetic field due to a certain current distribution of high symmetry without having to evaluate the complicated integrals that appear in the Biot-Savart law.

<u>H.W:</u> Starting from the integral form of Ampere's law deduce the corresponding differential form.

<u>Example (1):</u>

Using Ampere's law, deduce the *magnetic induction* at a distance $r = a$ from a *long straight wire*.

<u>Solution:</u> We have proved using Biot-Savart law the magnetic flux density is given as shown by equation (7-30). Here also we can prove that a similar result can be obtained by means of Ampere's law. So, Ampere's law is given as;



$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_o I \qquad . \quad . \quad . \quad (1)$$

Using the cylindrical coordinate, for the dot product in the left hand side;

$$\vec{B}\cdot d\vec{\ell} = |\vec{B}||d\vec{\ell}|cos\chi$$

$$= |\vec{B}|rd\varphi\hat{\varphi}$$

Where $\chi \approx 0, \rightarrow cos\chi \approx 1$

$\therefore eq(1) becomes;$

$$\int_0^{2\pi} |\vec{B}|rd\varphi = \mu_o I$$

$$|\vec{B}|r\varphi|_0^{2\pi} = \mu_o I$$

$$|\vec{B}| = \frac{\mu_o I}{2\pi r}$$

$$\boxed{\therefore \vec{B} = \frac{\mu_o I}{2\pi r}\hat{\varphi}}$$

Another procedure:

$$\oint_C B\hat{\varphi}\cdot(dr\,\hat{r} + rd\varphi\hat{\varphi} + dz\hat{k}) = \mu_o I$$

$$\oint_C B\,rd\varphi = \mu_o I$$

$$\int_0^{2\pi} Brd\varphi = \mu_o I$$

$$Br2\pi = \mu_o I$$

$$B = \frac{\mu_o I}{2\pi r}$$

$$\boxed{\vec{B} = \frac{\mu_o I}{2\pi a}\hat{\varphi}}$$

<u>Example (2)</u>:

Consider a coaxial cable consisting of a small center conductor of radius r_1 and a coaxial cylindrical outer cable conductor of radius r_2, as shown in figure 7-5. Assume that the two conductors carry equal total currents of magnitude I in opposite directions, Using Ampere's law, deduce the magnetic induction at a distance r from the center.

Solution:

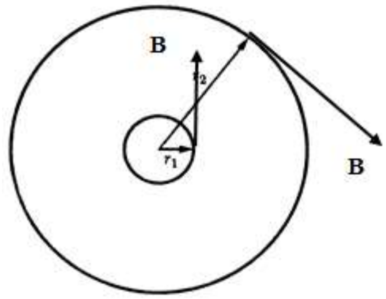


Figure 7-5: Cross section through a coaxial cable.

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_o I$$

$$B \oint_C d\ell = \mu_o I$$

$$B 2\pi r = \mu_o I$$

$$\boxed{B = \frac{\mu_o I}{2\pi r} \quad \{r_1 < r < r_2\}}$$

7-6 The magnetic vector potential

The calculation of electric fields was much simplified by the introduction of the electrostatic potential. The possibility of making this simplification resulted from the vanishing of the curl of the electric field ($\vec{\nabla} \times \vec{E} = 0$) in chapter 2.

The curl of the magnetic induction does not vanish; however, its divergence does (eq. $(7-*)$). Since the divergence of any curl is zero, it is reasonable to assume that the magnetic induction may be written;

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A} \quad . \ . \ . \ (7-37)}$$

Where A is called *magnetic vector potential* which given by the following expression;

$$\vec{A}(r_2) = \frac{\mu_o}{4\pi}\int_V \frac{\hat{J}(r_1)}{|\vec{r}_2 - \vec{r}_1|} dV_1 \qquad . . . (7-38)$$

Magnetic vector potential

The only other requirement placed on A is that, see equation (7-33);

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu_o \vec{J} \quad . . . (7-39)$$

Using the identity;

$$curl\ curl\ \vec{A} = grad\ div\ \vec{A} - \nabla^2 \vec{A} \quad . . . \quad (7-40)$$

Specifying that: $div\ A = 0$, yields

$$\nabla^2 \vec{A} = -\mu_o \vec{J} \quad . . . \quad (7-41)$$

Equation (6-41) called *Vector Poisson's Equation* in magnetostatic. Actually there are many way by means equation (6-38) could be derived, one of them are shown in the example below.

The Proof:

Derive the form of the magnetic vector potential – eq.(7-38).

Solution:

The *integral form of Biot-Savart law* (current density form) is;

$$\vec{B}(r_2) = \frac{\mu_o}{4\pi}\int_V \vec{J}(r_1) \times \frac{(\vec{r}_2 - \vec{r}_1)}{|(\vec{r}_2 - \vec{r}_1)|^3} dv_1 \quad . . . \quad (7-25)$$

we have

$$\frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} = -\vec{\nabla}_2 \frac{1}{|\vec{r}_2 - \vec{r}_1|}$$

$$\vec{B}(r_2) = \frac{\mu_o}{4\pi}\int_V \vec{J}(r_1) \times -\vec{\nabla}_2 \frac{1}{|\vec{r}_2 - \vec{r}_1|} dv_1$$

Where $\vec{\nabla}_2$ means that the differentiation is with respect to $\vec{r}_2$, while $\vec{J}(r_1)$ is with respect to r_1 ;

$$= -\frac{\mu_o}{4\pi}\int_V \vec{\nabla}_2 \times \frac{\vec{J}(r_1)}{|\vec{r}_2 - \vec{r}_1|} dv_1$$

Application of the identity;

$$\nabla \times \varphi\vec{A} = \varphi\,\vec{\nabla} \times \vec{A} - \vec{A} \times \vec{\nabla}\varphi$$

where $\varphi = \frac{1}{|\vec{r}_2 - \vec{r}_1|}, \vec{A} = \vec{J}(r_1)$

yields;

$$\vec{\nabla}_2 \times \frac{\vec{J}(r_1)}{|\vec{r}_2 - \vec{r}_1|} = \frac{1}{|\vec{r}_2 - \vec{r}_1|}\,\vec{\nabla}_2\,\vec{J}(r_1) - \vec{J}(r_1) \times \vec{\nabla}_2 \frac{1}{|\vec{r}_2 - \vec{r}_1|}$$

Since $J(r_1)$ does not depend on r_2 the first term will vanishes and so we have;

$$\vec{B}(r_2) = \vec{\nabla}_2 \times \left\{\frac{\mu_o}{4\pi}\int_V \frac{\vec{J}(r_1)}{|\vec{r}_2 - \vec{r}_1|}\, dv_1\right\}$$

Compare with equation ($\vec{B} = \vec{\nabla} \times \vec{A}$) yield;

$$\vec{A}(r_2) = \frac{\mu_o}{4\pi}\int_V \frac{\hat{J}(r_1)}{|\vec{r}_2 - \vec{r}_1|} dv_1$$

Which is exactly the equation (7-38).

<u>(7-7) Magnetic scalar potential:</u>

Using Ampere's law ($curl\,\vec{B} = \mu_o\vec{J}$) $div\vec{B} = \mu_0\vec{J}$, prove that the magnetic scalar potential U* satisfies the Laplace's equation

Differential form of Amp. law $\vec{\nabla} \times \vec{B}(r_2) = \mu_o\vec{J}(r_2)$ indicates that the curl of the magnetic induction is zero wherever the current density is zero. Thus the magnetic induction in such regions can be written as the gradient of scalar potential:

$$\boxed{\vec{B} = -\mu_o\vec{\nabla}U^*}$$

However, the divergence of $\vec{B}$ is also zero, which means that;

$$\boxed{\nabla \cdot \vec{B} = -\mu_o \nabla^2 U^* = 0}$$

U^* is called magnetic scalar potential, which is satisfies Laplace equation.

<u>H.W</u>

Compare between electrostatic and magnetostatic from point of view of the following laws:

Force, field (curl and divergence), potential (scalar and vector), Poisson and Laplace equations.

Problem 5.29 Use the results of Ex. 5.11 to find the field inside a uniformly charged sphere, of total charge Q and radius R, which is rotating at a constant angular velocity $\boldsymbol{\omega}$.

Problem 5.30

(a) Complete the proof of Theorem 2, Sect. 1.6.2. That is, show that any divergenceless vector field $\mathbf{F}$ can be written as the curl of a vector potential $\mathbf{A}$. What you have to do is find A_x, A_y, and A_z such that: (i) $\partial A_z/\partial y - \partial A_y/\partial z = F_x$; (ii) $\partial A_x/\partial z - \partial A_z/\partial x = F_y$; and (iii) $\partial A_y/\partial x - \partial A_x/\partial y = F_z$. Here's one way to do it: Pick $A_x = 0$, and solve (ii) and (iii) for A_y and A_z. Note that the "constants of integration" here are themselves functions of y and z—they're constant only with respect to x. Now plug these expressions into (i), and use the fact that $\nabla \cdot \mathbf{F} = 0$ to obtain

$$A_y = \int_0^x F_z(x', y, z)\, dx'; \quad A_z = \int_0^y F_x(0, y', z)\, dy' - \int_0^x F_y(x', y, z)\, dx'.$$

(b) By direct differentiation, check that the $\mathbf{A}$ you obtained in part (a) satisfies $\nabla \times \mathbf{A} = \mathbf{F}$. Is $\mathbf{A}$ divergenceless? [This was a very asymmetrical construction, and it would be surprising if it *were*—although we know that there *exists* a vector whose curl is $\mathbf{F}$ *and* whose divergence is zero.]

(c) As an example, let $\mathbf{F} = y\,\hat{\mathbf{x}} + z\,\hat{\mathbf{y}} + x\,\hat{\mathbf{z}}$. Calculate $\mathbf{A}$, and confirm that $\nabla \times \mathbf{A} = \mathbf{F}$. (For further discussion see Prob. 5.51.)

5.4.2 Summary; Magnetostatic Boundary Conditions

In Chapter 2, I drew a triangular diagram to summarize the relations among the three fundamental quantities of electrostatics: the charge density ρ, the electric field $\mathbf{E}$, and the potential V. A similar diagram can be constructed for magnetostatics (Fig. 5.48), relating

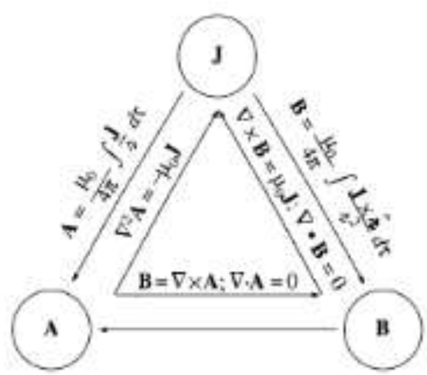


Figure 5.48