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Chapter One
VECTOR ANALYSIS

In the study of Electricity and Magnetism a great saving in complexity of notation may be accomplished by using the notation of vector analysis. However this is the purpose of this chapter in order to give a brief exposition of basic vector analysis and to provide the rather utilitarian knowledge of the field which is required for the treatment of electricity and magnetism.

1.1: Definitions

In the study of elementary physics several kinds of quantities have been encountered. In fact, they are mainly divided into two mainly kinds; which are *scalars* and *vectors*.

A scalar is a quantity which is completely characterized by its magnitude. Examples for this sort are; mass, time, volume … etc. A simple extension of the idea of scalar is a *scalar field*, which is defined as; *a function of position that is completely specified by its magnitude at all points in space*.

A vector quantity may be defined as; *a quantity which is completely characterized by its magnitude and direction.* As example for vector we cite position from a fixed point, velocity, acceleration, force … etc. The generalization to a *vector field* gives *a function of position which is completely specified by its magnitude and direction at all points in space*.

1.2: Vector Algebra

The operation of addition, subtraction, and multiplication familiar in the algebra of numbers are comparable, with suitable definition, of extension to algebra of vectors. The following definitions are fundamental:

1. Two vectors $\vec{A}$ and $\vec{B}$ are equal ($\vec{A} = \vec{B}$) if they have the same magnitude (length) and direction. fig.1.

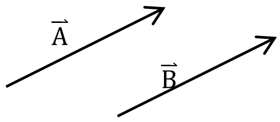


Fig.1

2. The negative of the vector $\vec{A}$ is written $-\vec{A}$, and has the same magnitude but opposite direction to $\vec{A}$. If $\vec{A} = \overrightarrow{BC}$, then $-\vec{A} = \overrightarrow{CB}$, fig.2.

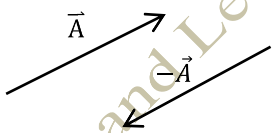


Fig.2

3. The graphical sum or resultant of vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C} = \vec{A} + \vec{B}$, which is formed by placing the initial of $\vec{B}$ on the terminal point of $\vec{A}$ and joining the initial point of $\vec{A}$ to the terminal point of $\vec{B}$. This definition is equivalent to the **"Triangle Rule"**, as in part (b) from the fig.3, or by **"The Parallelogram Law"**, as in part (c) from fig. 3.

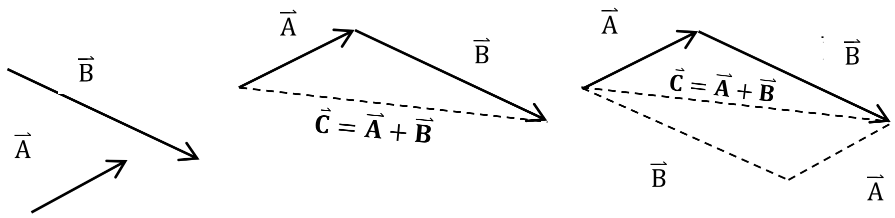


Fig. 3.

The magnitude or resultant vector can be determined by using either sine law or **cosine law**, fig.4, such that:

$$R^2 = A^2 + B^2 - 2ABcos\big(180 - (\alpha + \beta)\big)$$

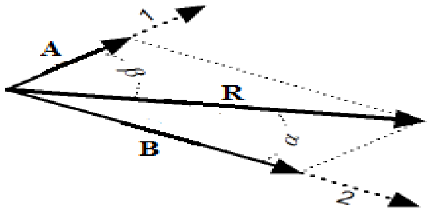


Fig. 4

In case of more than two vectors, **"Law of Polygon of Vectors"** is then used to calculate the resultant, fig. 4.

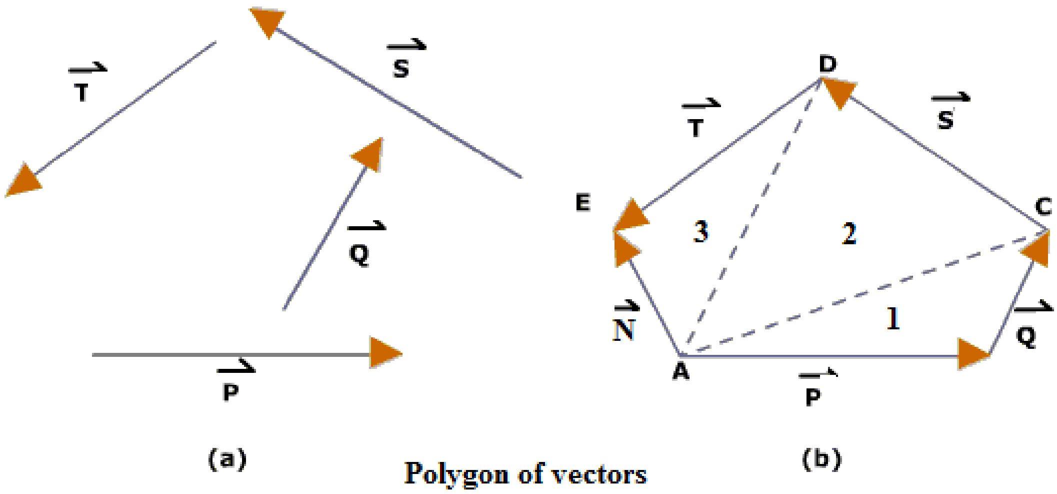


Fig. 5.

يتم وصل نهاية كل متجه مع بداية المتجه الثاني على الترتيب ثم يجري تقسيم الشكل الناتج الى مثلثات كل ضلعين فيها تمثل متجهين ومحصلتهما الضلع الثالث، ثم نتعامل مع هذه المحصلة كمتجه جديد يضاف الى المتجه الثالث وترسم محصلتهما، وهكذا لحين الانتهاء من كل المتجهات المعطاة، بحيث تكون محصلة الشكل السابق:

$$\vec{N} = \vec{P} + \vec{Q} + \vec{S} + \vec{T}$$

H.W:

1. Follow the last rule and calculate the resultant in details.

2. Find the graphical sum or resultant of vectors: $\vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$, that has shown in fig. 5.

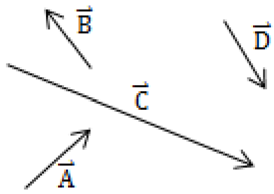


Fig. 6

4. The subtraction of vectors $\vec{A}$ and $\vec{B}$ is represented by the vector $\vec{C} = \vec{A} - \vec{B}$, where $\vec{C}$ is defined as that vector which deduced by adding $\vec{B}$ to $\vec{A}$. i.e. $\vec{A} + (-\vec{B})$. However, when $\vec{A} = \vec{B}$ then $\vec{C}$ is null or zero.

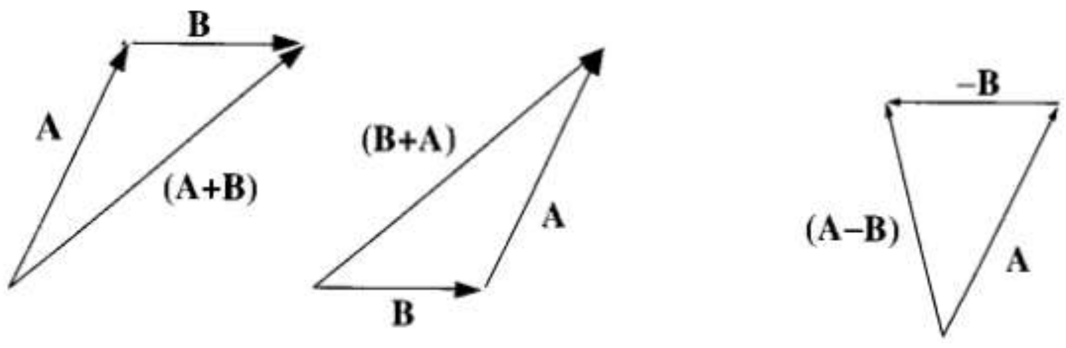
5. Multiplication of a vector $\vec{A}$ by a *positive scalar* m produce a vector magnitude but leaves the direction, while in case of *negative scalar* the resulting vector will be reversed.

● H.W.: When $m = 0$, what is the result of $m\vec{A}$ and what does it called?

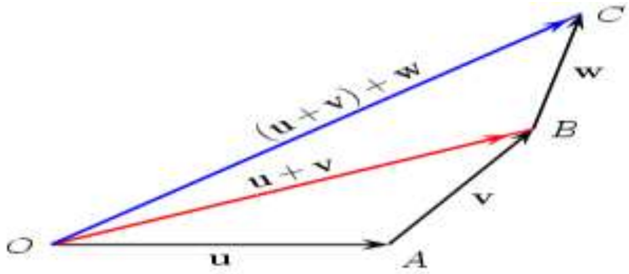
1.2.1: Laws of Vector Algebra

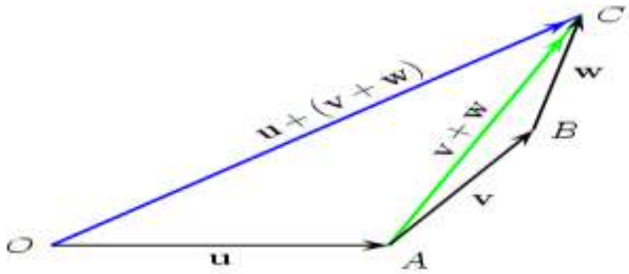
If m and n are scalar and $\vec{A}, \vec{B}$ and $\vec{C}$ are vectors, then;

1. $\vec{A} + \vec{B} = \vec{B} + \vec{A}$ (vector addition is commutative تبادلي as in graphs below)



2. $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$ (vector addition is associative)





3. $m(n\vec{A}) = n(m\vec{A})$ (associative law for *scalar* multiplication with vector)

4. $(m+n)\vec{A} = m\vec{A} + n\vec{A}$, also, $m(\vec{A}+\vec{B}) = m\vec{A} + m\vec{B}$ (distributive التوزيع scalar multiplication)

1.2.2 Components of a Vector (Grifiths p3)

Any vector $\vec{A}$ in a 3 – D. can be represented with initial point at the origin 0 of rectangular coordinates system, fig.7. Let (A_x, A_y, A_z) be the rectangular coordinates of the terminal point of vector $\vec{A}$ with initial point at 0.

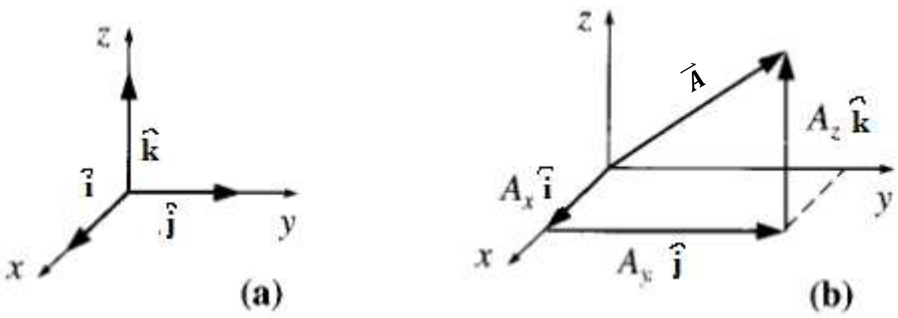


Fig. 7.

The vectors $A_x\hat{\imath}, A_y\hat{\jmath}$ and $A_z\hat{k}$ are called the rectangular component *vectors* or component vectors of $\vec{A}$ in the x, y and z directions, respectively. A_x, A_y and A_z are called the rectangular components or components of $\vec{A}$ in x, y and z directions, respectively, and $\hat{\imath}, \hat{\jmath}, \hat{k}$, are its unite vectors in those axis, respectively.

The sum or resultant of $A_x\vec{\imath}, A_y\vec{\jmath}$ and $A_z\vec{k}$ is the vector $\vec{A}$, so that we can write:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \quad \text{(Vector components)}$$

This tells us the *size* of the vector $\vec{A}$ in three directions. The magnitude of $\vec{A}$ is:

$$A = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad \text{(Vector magnitude)}$$

The *unite vector* (vector with magnitude one "$\hat{\imath}$ is called i-hat") for the vector $\vec{A}$ given as;

$$\hat{A} = \frac{\vec{A}}{A}$$

في مواضيع الكهربائية الساكنة والكهرومغناطيسية غالبا ما يتم التعامل مع شحنة في الفضاء، تكون هي المصدر، يراد قياس (كشف) احد الظواهر المتولده عنها، كالمجال، في نقطة اخرى في نفس الفضاء. ومن معرفة احداثيات موقع الشحنة واحداثيات موقع الرصد سيكون بالامكان حساب المسافة بين الموقعين باستخدام جبر المتجهات وما يسمى بـ"متجة الموقع او نصف القطر $\vec{r}$". عادة يتم اعتماد نقطة الاصل كمرجع لاي موقع في الفضاء.

In particular, the *position* vector or radius vector $\vec{r}$ from (x, y, z) to the origin is written:

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \quad \text{(Position vector)}$$

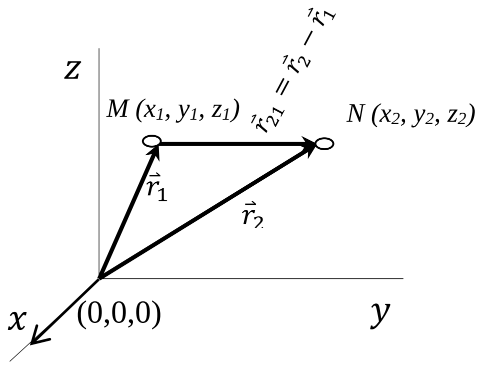
with its magnitude: $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

and its direction or *unite vector* is: $\hat{r} = \frac{\vec{r}}{r}$

Example:

Prove that the vector directed from *M*(x_2, y_2, z_2) to *N*(x_1, y_1, z_1) is given by; $(x_1 - x_2)\hat{\imath} + (y_1 - y_2)\hat{\jmath} + (z_1 - z_2)\hat{k}$.

Solution:



$\vec{r}_1 = (x_1 - 0)\hat{\imath} + (y_1 - 0)\hat{\jmath} + (z_1 - 0)\hat{k}$

$\vec{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$

The two vectors resultant (triangle rule) is: $\vec{r}_1 = \vec{r}_2 + \vec{r}_{21}$

$\vec{r}_{21} = \vec{r}_2 - \vec{r}_1$

$\vec{r}_{21} = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k} - (x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k})$

$\vec{r}_{21} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$

H.W.:

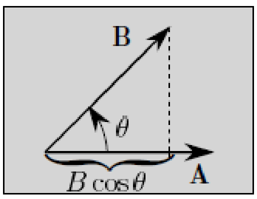
1. What will be the result if the vector is directed from N to M?

2. Given that $\vec{r}_1 = 3\hat{\imath} + 5\hat{\jmath} + 4\hat{k}$ and $\vec{r}_2 = \hat{\imath} + 3\hat{\jmath} + \hat{k}$, what will be the values of $\vec{r}_1$ and $\vec{r}_2$?

1.2.3 Dot (Scalar) Product

The dot or scalar product of two vectors $\vec{A}$ and $\vec{B}$ denoted by $\vec{A}.\vec{B}$ is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ times the *cosine* of the angle between them (when constructing as a tail with tail); i.e.

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|cos\theta \qquad 0 \le \theta \le \pi \qquad . \quad . \quad . \ (\#)$$



$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$

ان معاملات الطرف الايمن من المعادلة اعلاه جميعا قيم عددية لذلك فان الناتج هو عددي

Note that $\vec{A}.\vec{B}$ is a scalar and not vector. The following rules are valid:

1. $\vec{A}.\vec{B} = \vec{B}.\vec{A}$ *commutative* law for dot product.
2. $\vec{A}.(\vec{B} + \vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$ *distributive* law = = = .
3. $m(\vec{A}.\vec{B}) = (m\vec{A}).\vec{B} = \vec{A}.(m\vec{B}) = (\vec{A}.\vec{B})m$, where: m is a scalar.
4. $\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1, \hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0$ the angle between each two similar unite vectors is 0^o, while it is 90^o for those who are different.
5. H.W:

If $\vec{A} = A_x\hat{\imath} + A_y\hat{\imath} + A_z\hat{k}, \vec{B} = B_x\hat{\imath} + B_y\hat{\imath} + B_z\hat{k}$, then:

$$\vec{A}.\vec{B} = A_x.B_x + A_y\,B_y + A_zB_z$$

a) What is the difference between this formula and the formula (#)?

b) Find: $A^2$and B^2, where: $A^2 = \vec{A}.\vec{A}, also: B^2 = \vec{B}.\vec{B}$.

6. H.W:

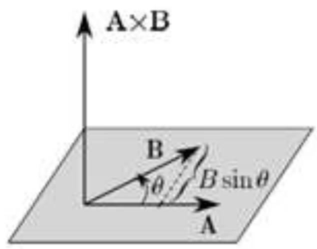
If $\vec{A}.\vec{B} = 0$, and $\vec{A}$ and $\vec{B}$ are not null vectors, then proof that $\vec{A}$ and $\vec{B}$ are perpendicular.

1.2.4 Cross or Vector Product

The cross or vector product of $\vec{A}$ and $\vec{B}$ is a vector $\vec{A} \times \vec{B}$. The magnitude of $\vec{A} \times \vec{B}$ is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ times the *sine* of the angle between them. The direction of the vector $\vec{A} \times \vec{B}$ is perpendicular to the plane of $\vec{A}$ and $\vec{B}$ and such that $\vec{A}$, $\vec{B}$ and $\vec{C}$ from a right-handed system. In symbols, geometric definition given as;

$$\vec{A} \times \vec{B} = (|\vec{A}||\vec{B}|sin\theta)\hat{n} \qquad 0 \le \theta \le \pi$$

$$where: \quad |\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|sin\theta$$



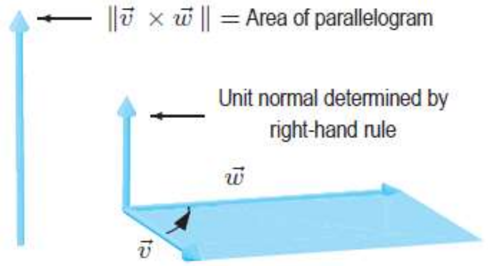


Figure 13.36: Area of parallelogram = $\|\vec v \times \vec w\|$

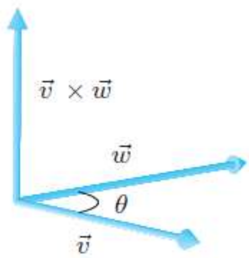
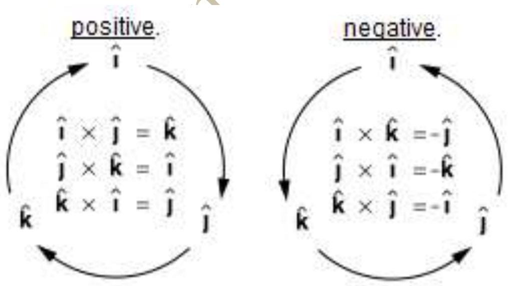


Figure 13.37: The cross product $\vec v \times \vec w$

where $\hat u$ is a unite vector indicating the direction of $\vec A \times \vec B$. This formula gives both the magnitude and direction. The following rules are valid:

1. $\vec A \times \vec B = -\vec B \times \vec A$ (anti-commutative cross product)
2. $\vec A \times (\vec B + C) = \vec A \times \vec B + \vec A \times C$ (*distributive* law = = =)
3. $m(\vec A \times \vec B) = (m\vec A) \times \vec B = \vec A \times (m\vec B) = (\vec A \times \vec B)m$, where: m is a scalar.
4. $\hat\imath \times \hat\imath = \hat\jmath \times \hat\jmath = \hat k \times \hat k = 0,\ \hat\imath \times \hat\jmath = \hat k,\ \ \hat\jmath \times \hat k = \hat\imath, \hat k \times \hat\imath = \hat\jmath$

where the result depends on the product direction (angle).



5. If $\vec A = A_x\hat\imath + A_y\hat\imath + A_z\hat k$, $\vec B = B_x\hat\imath + B_y\hat\imath + B_z\hat k$, Then: Algebraic definition for $\vec A \times \vec B$ is;

$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \boldsymbol{A}_x & \boldsymbol{A}_y & \boldsymbol{A}_z \\ \boldsymbol{B}_x & \boldsymbol{B}_y & \boldsymbol{B}_z \end{vmatrix}$$

6. $|\vec A \times \vec B|$ the area of *parallelogram* (= area of rectangular) with sides $\vec A$ and $\vec B$.

H.W:

7. Give the Physical meaning of $|\vec{A} \times \vec{B}| = 0$ when $\vec{A} \neq \vec{B}$ and both $\vec{A}$ and $\vec{B}$ are not null.

Example: Prove that $|\vec{A} \times \vec{B}|$ represents the area of a parallelogram with sides $\vec{A}$ and $\vec{B}$.

Solution:

$$\boxed{Area\ of\ parallelogram = Height \times Base}$$

i.e.

$$Area = h \cdot |\vec{A}|$$

$$= |\vec{B}| sin\theta \cdot |\vec{A}|$$

$$= |\vec{B}||\vec{A}| sin\theta$$

$$\therefore Area = |\vec{A} \times \vec{B}|$$

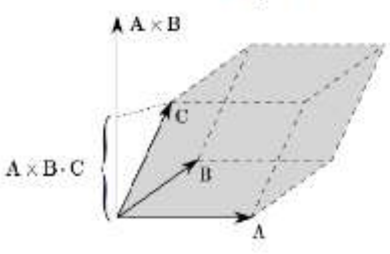
H.W:

1) If $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ what about $|\vec{A} \times \vec{B}| \overset{?}{=} |\vec{B} \times \vec{A}|$.

2) Given $\vec{A} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\vec{B} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$ find $\vec{A} \times \vec{B}$.

3) Find the area of the parallelogram with edges $\vec{v} = 2\vec{i} + \vec{j} - 3\vec{k}$ and $\vec{w} = \vec{i} + 3\vec{j} + 2\vec{k}$.

1.2.5 Triple Product



A×B · C = ± volume of
A, B, C parallelepiped

Area of base of parallelepiped $= \|\vec{b} \times \vec{c}\|$.

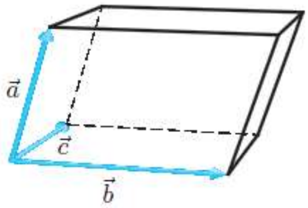


Figure 13.39: Volume of a Parallelepiped

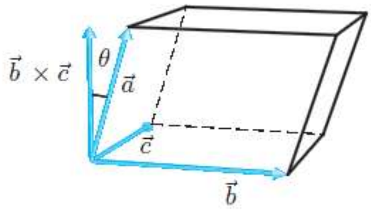


Figure 13.40: The vectors $\vec{a}, \vec{b}, \vec{c}$ are called a right-handed set

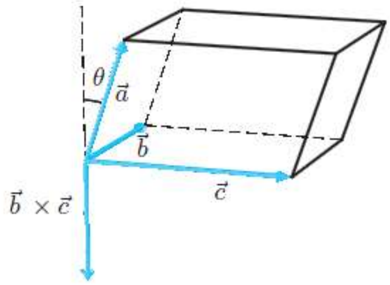


Figure 13.41: The vectors $\vec{a}, \vec{b}, \vec{c}$ are called a left-handed set

The following rules are valid;

1. $(\vec{A}.\vec{B})\vec{C} \neq \vec{A}(\vec{B}.\vec{C})$
2. $\vec{A}.(\vec{B} \times \vec{C}) = \vec{B}.(\vec{C} \times \vec{A}) = \vec{C}.(\vec{A} \times \vec{B})$ = the volume of a parallelepiped having $\vec{A}, \vec{B}$ and $\vec{C}$ as edges, where;

$$\vec{A}.(\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

3. $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \times \vec{C}$, associative law for cross product.
4. $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$

   $(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A}.\vec{C})\vec{B} - (\vec{B}.\vec{C})\vec{A}$

Notes:

a. The product $\vec{A}.(\vec{B} \times \vec{C})$ is sometimes called the *scalar triple product* or *box* product; why?

b. The product $\vec{A} \times (\vec{B} \times \vec{C})$ is called the *vector product*.

H.W:

Why the parentheses in the box products $\vec{A}.(\vec{B} \times \vec{C})$ can be omitted, while, it is note valid for the vector triple product; $\vec{A} \times (\vec{B} \times \vec{C})$?

Example:

Prove that $\vec{A}.\left(\vec{B}\times\vec{C}\right)$ is in absolute value equal to the volume of parallelepiped with sides $\vec{A}$, $\vec{B}$ and $\vec{C}$.

Solution:

Volume = height (h) × area parallelogram

$$Volume = |\vec{A}|\, cos\theta\ \times (|\vec{B}\times\vec{C}|)$$

$$Volume = |\vec{A}|\ \cdot \hat{n}\, (|\vec{B}\times\vec{C}|)$$

$$Volume = |\vec{A}|\ \cdot\ (|\vec{B}\times\vec{C}|\hat{n})$$

$$Volume = |\vec{A}\ \cdot\ \vec{B}\times\vec{C}|$$

1.3 Derivative

Assume we have a function of one variable say $f(x)$, the derivative $\frac{df}{dx}$ will measure how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount dx, i.e.

$$df = \left(\frac{df}{dx}\right)dx \quad ...\ (1\text{-}1)$$

In other words, when we change x by an amount dx, then f will change by an amount df. Hence, the derivative being a proportionality factor. For example in the figure 6-a below, the function varies slowly with x and hence the corresponding derivative will be small.

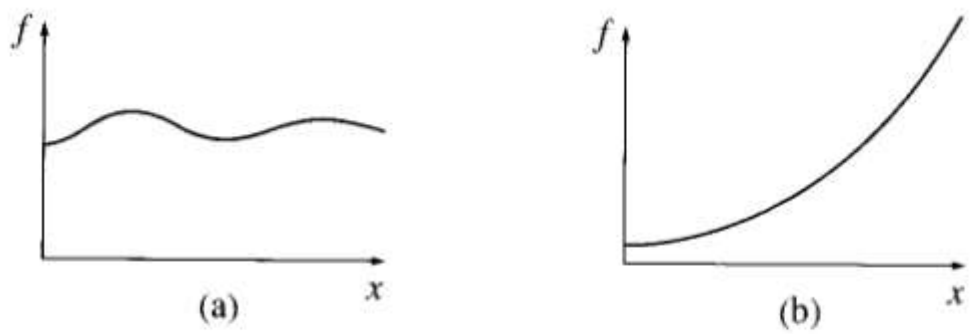


Fig.6

In part (b) from the previous fig., f increases rapidly with x and the derivative is large as we move away from $x = 0$. The geometrical interpretation of the derivative df/dx is that it is the slope of the graph of $f$versus x.

1.3.1 The Gradient ($\vec{\nabla}$)

Let us assume now that the function f of a three variables say (x, y, z) instead of one (x). Here again, the derivative measures how fast the function f varies if we move a little distance. But the situation now is more complicated because it depends on what direction we move. Correspondingly, the theorem on partial derivative may describe this situation, i.e.

$$df = \left(\frac{df}{dx}\right)dx + \left(\frac{df}{dy}\right)dy + \left(\frac{df}{dz}\right)dz \quad ... \ (1-2)$$

This equation state that how the function f changes when we alter all three variables by the infinitesimal amounts dx, dy and dz. However, this equation reminiscent of dot product;

$$df = \left\{\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right\}.\left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right)$$

$$= \vec{\nabla}f.d\vec{l} \quad ... (1-3)$$

Where;

$$\boxed{\vec{\nabla}f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}}$$

is the *gradient* of f. Actually $\vec{\nabla}f$ is a *vector quantity* with three components, in comparing with f which was a scalar function, one dimensional with any number of variables.

The geometrical interpretation of the gradient may thought as in the following. The gradient like any vector has a magnitude and direction. To determine its geometrical meaning lets rewrite the dot product in equation (1 – 3) in the following form;

$$df = \vec{\nabla}f.d\vec{l} = |\vec{\nabla}f|.|d\vec{l}|cos\theta$$

where θ is the angle between $\vec{\nabla} f$ and $d\vec{l}$. Now by fixing the magnitude $|d\vec{l}|$ and search around in various directions (that is vary θ), one may found that the maximum change in f evidentially occurs when $\theta = 0$, $i.e.$ $(cos\theta = 1)$. This mean, however, for a fixed distance $|d\vec{l}|$, df is greatest when $d\vec{l}$ move in the same direction as $\vec{\nabla} f$. Thus;

*"the gradient $\vec{\nabla} f$ points in the direction of **maximum** increase of the function and the magnitude $|\vec{\nabla} f|$ gives the **slope** (rate of increase) along this maximal direction".*

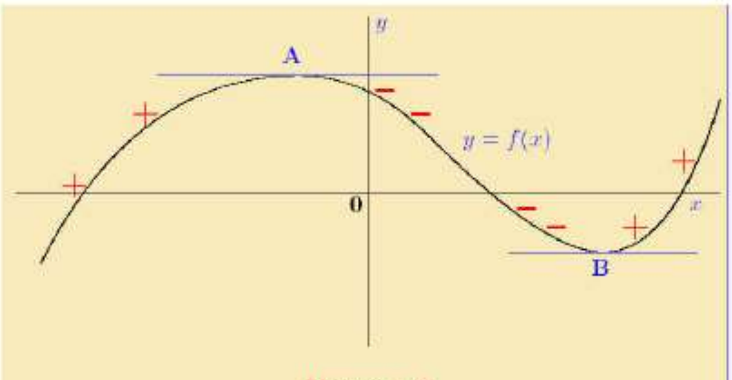


Diagram 2

Further observations on the gradients of tangents to the curve are:

to the *left* of A the gradients are positive (+)
between A and B the gradients are negative (−)
to the *right* of B the gradients are positive (+)

where A and B are stationary points.

H.W: What would be the meaning of $\vec{\nabla} f = 0$?

Example:

Find the gradient for the magnitude of the position vector: $|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$.

Solution:

$$\vec{\nabla}r = \frac{\partial r}{\partial x}\hat{\imath} + \frac{\partial r}{\partial y}\hat{\jmath} + \frac{\partial r}{\partial z}\hat{k}$$

$$\vec{\nabla}r = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}\hat{\imath} + \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}\hat{\jmath} + \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

$$\vec{\nabla}r = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{r} = \hat{r} \equiv \hat{n}$$

This result means that the distance from the origin increases most rapidly in the radial direction and the rate of increase is 1.

H.W:

1. Find the gradient of the following functions;
   a. $f(x,y,z) = x^2 + y^3 + z^4$
   b. $f(x,y,z) = x^2y^2z^2$
2. Let $\vec{r}$ be the separation vector from a fixed point $(\acute{x},\acute{y},\acute{z})$ to the point (x,y,z) and r be its length. Show that;
   a. $\vec{\nabla}(r^2) = 2\vec{r}$
   b. $\vec{\nabla}(1/r) = -\vec{r}/r^3$
   c. $\vec{\nabla}\frac{1}{r^3} = -\frac{3\vec{r}}{r^5}$

Remarks:

1. The del operator is not a vector in the usual sense because it is without specific meaning until we provide it with a function to act upon.

2. An ordinary vector $\vec{A}$ can multiply in three ways;
   a. $a\vec{A}$ (multiplication by a scalar a)
   b. $\vec{A}.\vec{B}$ (multiplication by another vector via dot product)
   c. $\vec{A} \times \vec{B}$ (multiplication by another vector via cross product)

Accordingly, there are three ways the operator $\vec{\nabla}$ can act namely;

   a. On a scalar function f : $\vec{\nabla}f \rightarrow gradient.$

b. On a vector function $\vec{V}$ via the **dot** product: $\vec{\nabla}.\vec{V} \rightarrow divergence.$

c. On a vector function $\vec{V}$ via the **cross** product: $\vec{\nabla}\times\vec{V} \rightarrow curl.$

1. 3. 2 The Divergence ($\vec{\nabla}\cdot$)

The divergence of a vector $\vec{V}$ is written $\vec{\nabla}.\vec{V}$ and defined as;

$$\vec{\nabla}.\vec{V} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\left(V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k}\right)$$

$$\boxed{\vec{\nabla}.\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}}$$

It can be seen that, the divergence of a vector function $\vec{V}$ is a scalar quantity. The geometrical interpretation for $\vec{\nabla}.\vec{V}$ is that *it measure how much the vector $\vec{V}$ is spread out (divergences) from the point in question.* For example the vector function in part (a) from the following figure has a large (positive) divergence. In fact when the arrows are pointed in, it would be a large negative divergence. Anyway, the function in part (b) from the same figure has a zero divergence, while that in part (c) from it has a positive divergence.

"Diverge" means to move away from, which may help you, remember that divergence is the rate of flux expansion (positive div) or contraction (negative div).

The divergence of a vector $\vec{F}$ is a derivative that measure how much this vector will spread or diverged from point to point in space. However, this process may defined as follows;

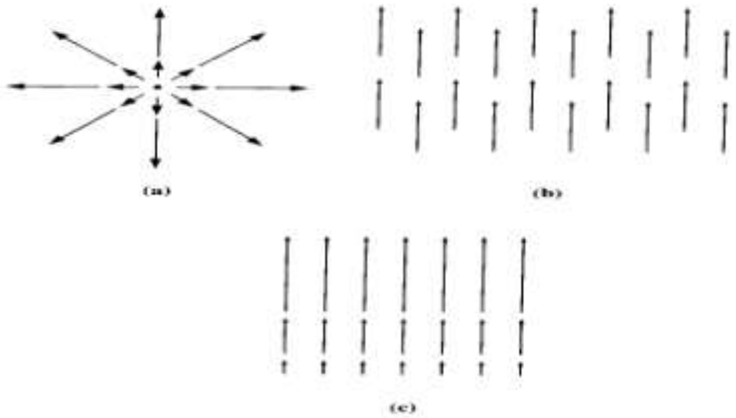


Fig.

" the divergence of a vector is the limit of its surface integral per unit volume as the volume enclosed by the surface goes to zero. i.e.

$$div\ \vec{F} = \vec{\nabla}\cdot\vec{F} = \lim_{V\to 0}\frac{1}{V}\oint_S \vec{F}.\hat{n}da$$

Example: Calculate the divergence of the following vector functions;

1. $\vec{\nabla}.\vec{V_a} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$
2. $\vec{V_b} = \hat{k}$
3. $\vec{V_c} = z\hat{k}$

Solution:

1. $\vec{\nabla}.\vec{V_a} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$
2. $\vec{\nabla}.\vec{V_b} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0 + 0 + 0 = 0$
3. $\vec{\nabla}.\vec{V_c} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 0 + 0 + 1 = 1$

H.W:

1. Calculate the divergence of the following vectors;

a. $\vec{V_a} = x^2\hat{\imath} + 3xz^2\hat{\jmath} + 3xz\hat{k}$

b. $\vec{V_b} = xy\hat{\imath} + 2yz\hat{\jmath} + 3zx\hat{k}$

c. $\vec{V_c} = y^2\hat{\imath} + (2xy + z^2)\hat{\jmath} + 2yz\hat{k}$

2. Find the divergence of the vector function; $\vec{V} = \frac{\hat{r}}{r^2}$ and explain the result.

1. 3. 3 The Curl ($\vec{\nabla}\times$)

The curl of a vector $\vec{V}$ is written $\vec{\nabla}\times\vec{V}$ or ($curl\ \vec{V}$) and defined as;

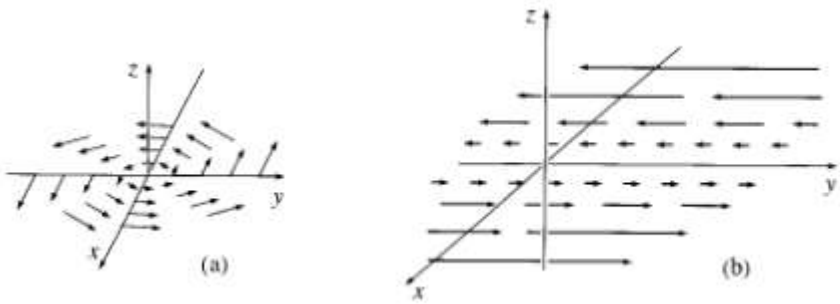
$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

$$\vec{\nabla}\times\vec{V} = \hat{\imath}\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)+\hat{\jmath}\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)+\hat{k}\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)$$

It is seen that the *curl* of a vector function $\vec{V}$, is like any cross product, is a vector. Actually one cannot have a curl for a scalar quantity since it has no direction and hence the result being of a meaningless.

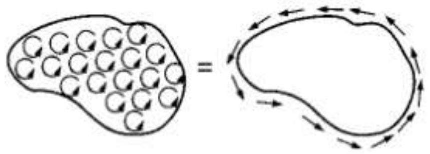
$$\begin{aligned}\nabla\times\mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} \\ &= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\end{aligned}$$

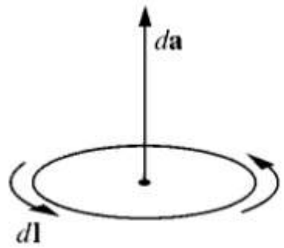
The geometrical interpretation for the curl process ($\vec{\nabla}\times\vec{V}$) is that *it is a differential process which measure how much the vector $\vec{V}$ will curl (rotate) around the point in question.* Thus, all vectors shown in previous figure have a zero curl. Whereas the function shown in the following figure have a curl pointing in the z-direction.



The curl operator is also a derivative process by which the rotation of a vector is measured. However, it is defined as; " The curl of a vector is the limit of the ratio of the integral of its cross product with the outward drawn normal, over a closed surface, to the volume enclosed by the surface as the volume goes to zero". i.e.

$$curl\,\vec{F} = \vec{\nabla} \times \vec{F} = \lim_{V\to 0} \frac{1}{V} \oint_S \hat{n}\times\vec{F}\ da \ldots \quad (1-7)$$





H.W:

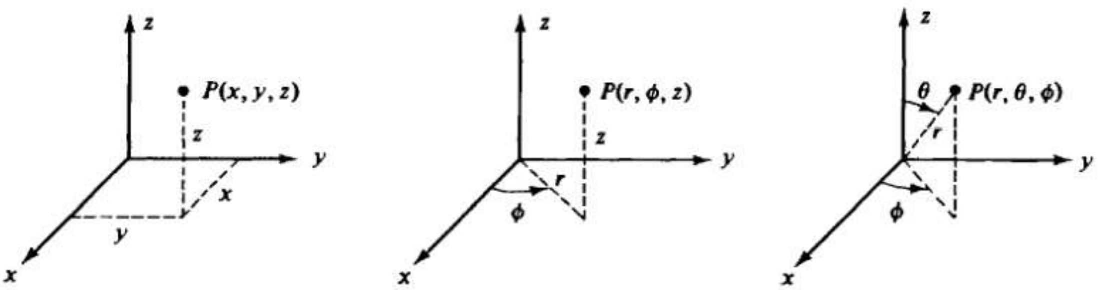
1. Calculate the curl of the following vectors;

a. $\vec{V}_a = -y\hat{\imath} + x\hat{\jmath}$

b. $\vec{V}_b = xy\hat{\imath}$

1. 4: Coordinate systems

Actually different kinds of coordinates systems usually used in physics. The most often used ones are *Cartesian*, *Cylindrical*, and *Spherical*. A point P may described by these three coordinates as shown in the figure below;



The order of specifying the coordinates is important an should carefully be followed. The angle φ is the same in both cylindrical an spherical systems. But in the order of the coordinates, φ appears in the second position in cylindrical (r, φ, z) and third position in spherical (r, θ, φ). The same sample r is used in both cylindrical and spherical coordinates but for two quit different things. In cylindrical coordinate r measure the distance from z-axis to a plane normal to the xy-plane. While in the spherical system r measure the distance from the origin to the point $P(r, \theta, \varphi)$.

Remarks:

1) The domain of the coordinates are as follows;

Cartesian	Cylindrical	Spherical
$-\infty \leq x \leq +\infty$	$0 \leq r \leq \infty$	$0 \leq r \leq \infty$
$-\infty \leq y \leq +\infty$	$0 \leq \varphi \leq 2\pi$	$0 \leq \theta \leq \pi$
$-\infty \leq z \leq +\infty$	$-\infty \leq z \leq +\infty$	$0 \leq \varphi \leq 2\pi$

2) The transformation equations from cylindrical to Cartesian coordinates are;

$$x = r\, cos\varphi$$
$$y = r\, sin\varphi$$
$$z = z$$

3) The transformation equations from spherical to Cartesian coordinates are;

$$x = r\, sin\theta\, cos\varphi$$
$$y = r\, sin\theta\, sin\varphi$$
$$z = r\, cos\theta$$

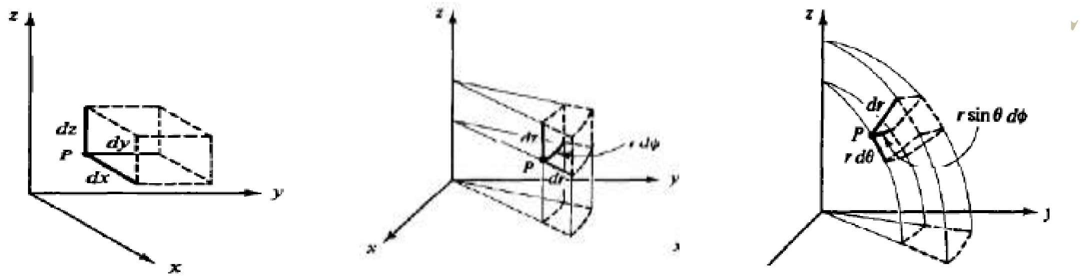
4) The components forms of a vector in three systems are;

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \qquad (Carteisan)$$
$$\vec{A} = A_r\hat{r} + A_\varphi\hat{\varphi} + A_z\hat{k} \qquad (Cylindrical)$$
$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi} \qquad (Spherical)$$

Question: What will happened when a point $P(x, y, z)$ is expanded to $(x + dx, y + dy, z + dz)$ or $(r + dr, \varphi + d\varphi, z + dz)$ or $(r + dr, \theta + d\theta, \varphi + d\varphi)$?

Answer:

As show in the figures;



1) A differential volume dV is formed;

$$dV = dxdydz \qquad \text{(Cartesian)}$$

$$dV = rdrd\varphi dz \qquad \text{(Cylindrical)}$$

$$dV = r^2 sin\theta drd\theta d\varphi \qquad \text{(Spherical)}$$

2) A differential area dS is formed;

$$ds = dxdy, = dxdz, = dydz \qquad \text{(Cartesian)}$$

$$ds = drdz, = rdrd\varphi, = rd\varphi dz \qquad \text{(Cylindrical)}$$

$$ds = rdrd\theta, = r^2 sin\theta d\theta d\varphi, = rsin\theta drd\varphi \qquad \text{(Spherical)}$$

3) A differential line $d\ell$ is formed;

$$d\vec{l} = dx\hat{\imath} + dy\hat{\jmath} + dz\vec{k} \qquad \text{(Cartesian)}$$

$$d\vec{l} = dr\hat{r} + rd\varphi\hat{\varphi} + dz\hat{z} \qquad \text{(Cylindrical)}$$

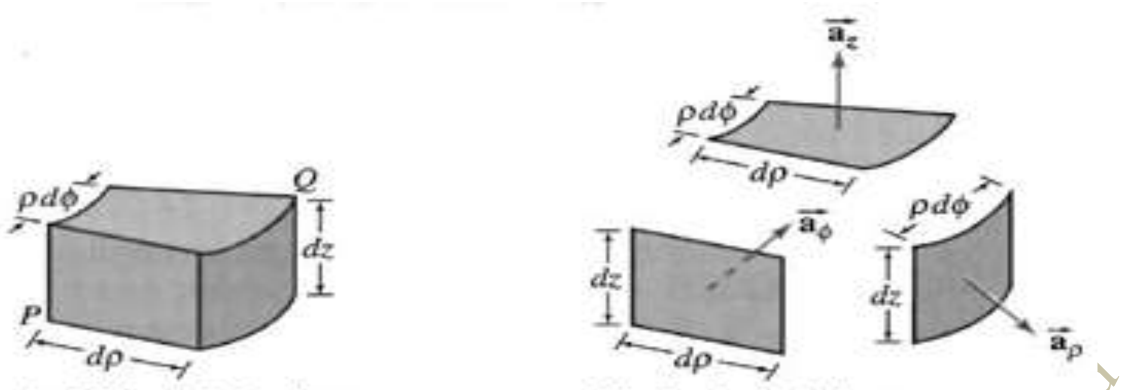
$$d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + rsin\theta d\varphi\hat{\varphi} \qquad \text{(Spherical)}$$

However;

$$dl^2 = dx^2 + dy^2 + dz^2 \qquad \text{(Cartesian)}$$

$$dl^2 = dr^2 + r^2 d\varphi^2 + dz^2 \qquad \text{(Cylindrical)}$$

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2 \qquad \text{(Spherical)}$$



a) Differential volume b) Exploded view

H.W.:

I) What are the forms of $\vec{\nabla}$ and $\vec{\nabla}^2$in the three coordinates systems?

II) Prove the following identities (table "1-1" in the text book):

(متطابقات للحفظ)

A. Grad.:

1. $\vec{\nabla}(V+V)=\vec{\nabla}V+\vec{\nabla}V$
2. $\vec{\nabla}(\vec{A}.\vec{B})=(\vec{B}.\vec{\nabla})\vec{A}+(\vec{A}.\vec{\nabla})\vec{B}+\vec{B}\times(\vec{\nabla}\times\vec{A})+\vec{A}\times((\vec{\nabla}\times\vec{B})$
3. $\vec{\nabla}fg=f\vec{\nabla}g+g\vec{\nabla}f$

B. Div.:

1. $\vec{\nabla}.(\vec{A}+\vec{B})=\vec{\nabla}.\vec{A}+\vec{\nabla}.\vec{B}$
2. $\vec{\nabla}.(V\vec{A})=(\vec{\nabla}V).\vec{A}+V(\vec{\nabla}.\vec{A})$
3. $\vec{\nabla}.(\vec{A}\times\vec{B})=\vec{B}.(\vec{\nabla}\times\vec{A})-\vec{A}.(\vec{\nabla}\times\vec{B})$
4. $\vec{\nabla}.(\vec{\nabla}V)=\nabla^2V$
5. $\vec{\nabla}.(\vec{\nabla}\times\vec{A})=0$ (the divergence of the curl of $\vec{A}$ is zero)

C. Curl:

1. $\vec{\nabla}\times(\vec{A}+\vec{B})=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$
2. $\vec{\nabla}\times(V\vec{A})=V\vec{\nabla}\times\vec{A}-\vec{A}\times(\vec{\nabla}V)=\vec{\nabla}V\times\vec{A}+V(\vec{\nabla}\times\vec{A})$
3. $\vec{\nabla}\times(\vec{A}\times\vec{B})=(\vec{B}.\vec{\nabla})\vec{A}-\vec{B}(\vec{\nabla}.\vec{A})-(\vec{A}.\vec{\nabla})\vec{B}+\vec{A}(\vec{\nabla}.\vec{B})$

4. $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A}$
5. $\vec{\nabla} \times (\vec{\nabla}V) = 0$ (the curl of gradient of V is zero)
6. $\vec{\nabla} \times \vec{A} = 0$, when $\vec{A}$ be a constant vector.

1 - 5 Vector Integration

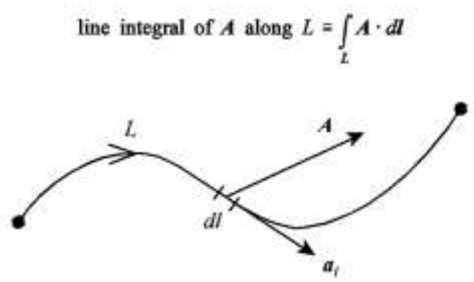
In fact three types of integrals one may consider namely; linear, surface and volume integrals according to the nature of the differential element appearing in the integral. The integral may be either *vector* or *scalar*.

1. Line integral

These are integrals along lines or curves, open or closed($\int$ *or* $\oint$). The line integral of a vector $\vec{F}$ is written;

$$\int_{c\,a}^{b} \vec{F}.d\vec{l} \qquad \ldots (1-4)$$

Where c is the curve along which the integration is performed, a and b are the initial and final positions on the curve, and $d\vec{l}$ is an infinitesimal vector displacement along the curve c.



Since $\vec{F}.d\vec{l}$ is a scalar quantity so the line integral is a scalar quantity. the definition of line integral can be thought as follows; if the segment of the curve c is divided into a large number of small increments $\Delta\vec{l}_i$. For each increment there will be a corresponding value of the vector $\vec{F}$, i.e. $\vec{F}_i$.

Then the line integral is defined as the limit of the sum for the scalar product $(\vec{F}_i . \Delta l_i)$ as the number of increments goes to infinity in such a way that each increment goes to zero, i.e.;

$$\int_{\substack{c \\ a}}^{b} \vec{F}.d\vec{l} = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}_i . \Delta\vec{l}_i$$

Remarks:

1. Line integral depends on the end points a and b.
2. Line integral depends on curve along which the integration is performed.
3. The line integral around a closed curve is denoted by;

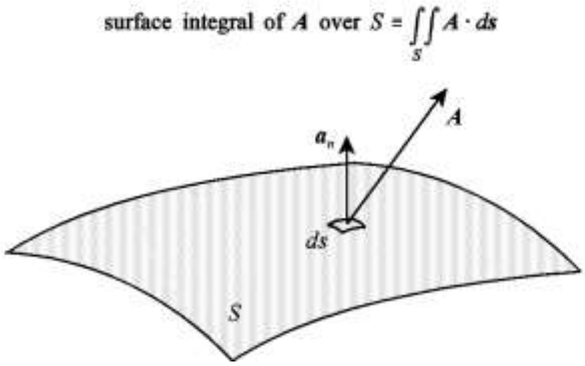
$$\oint_c \vec{F}.d\vec{l}$$

which may or may not be zero.

2. <u>Surface Integral:</u>

The surface integral of a vector $\vec{F}$ is defined as the integral of the scalar product of $\vec{F}$ with the unit vector that normal on infinitesimal area da of the surface S. i.e.

$$\int_S \vec{F}.\hat{n}\, da$$



Remarks:

1. The surface integral of $\vec{F}$ over a closed surface s is denoted by;

$$\oint_s \vec{F}.\hat{n}\,da$$

2. The surface integral is a scalar quantity.
3. The surface integral usually depends on the surface S.

3. Volume Integral:

The volume integral for a vector $\vec{F}$ and a scalar φ over a volume v is defined as;

$$\vec{K} = \int_V \vec{F}dv \text{ and } J = \int_V \varphi dv$$

Where $\vec{K}$ is a vector and J is a scalar.

1 - 6 Divergence Theorem

The volume integral of the divergence of a vector over a volume V is equal to the surface integral of the normal component of that vector over the surface S bounding it, such that:

$$\int_V \vec{\nabla}.\vec{F}\,dV = \oint_s \vec{F}.\hat{n}\,da \quad ...\ (1-6)$$

أن التكامل الحجمي لتباعد متجه على حجم ما يساوي التكامل السطحي للمتجه نفسه على المساحة المحيطة بالحجم نفسه، او ان الفيض الكلي لمجال متجه $\vec{A}$ خارج من سطح مغلق S يساوي التكامل الحجمي لتباعد هذا المتجه.

1 - 7 The Stokes' Theorem

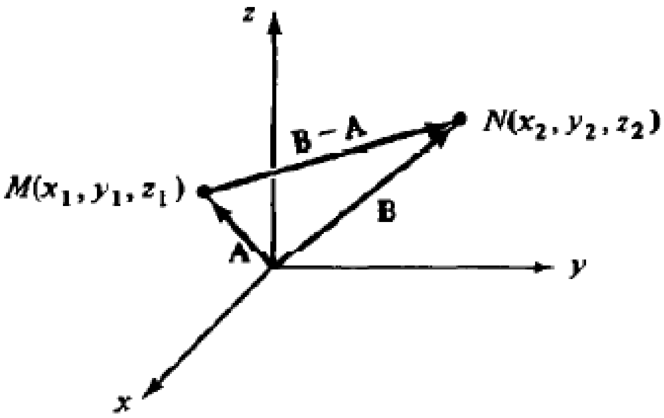
The line integral of a vector around a closed curve is equal to the surface integral of the normal component of its curl over the surface S bounded by that curve:

$$\int_s \vec{\nabla}\times\vec{F}\cdot\hat{n}\,da = \oint_c \vec{F}.d\vec{l} \quad ...\ (1-7)$$

تكامل الالتفاف لمتجه على سطح ما = التكامل الخطي للمتجه نفسه على المسار الخطي المحيط بتلك المساحة، أي أن التكامل الخطي المغلق لمجال متجه حول مسار مغلق يساوي التكامل السطحي لالتفاف هذا المتجه في الاتجاه العمودي على السطح المحاط بالمسار المغلق.

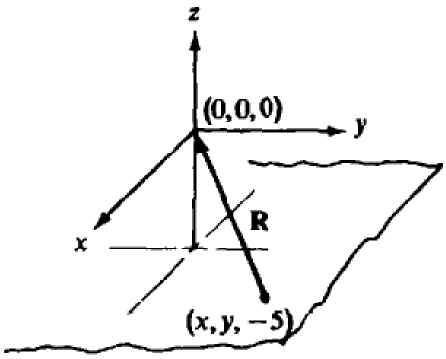
Solved Problems

1. Show that the vector directed from $M(x_1, y_1, z_1)$ to $N(x_2, y_2, z_3)$ in fig. is given by: $(x_2 - x_1)\hat{\imath} - (y_2 - y_1)\hat{\jmath} - (z_2 - z_1)\hat{k}$

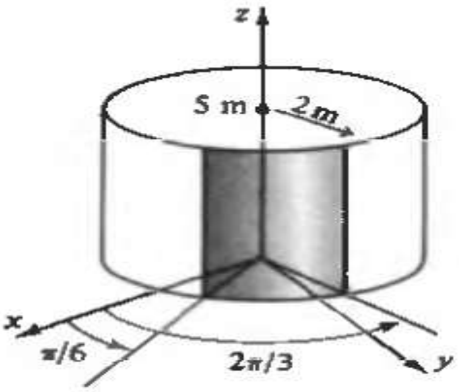


2. Find the vector $\vec{A}$ directed from $(2, -4, 1)$ to $(0, -2, 0)$ in Cartesian coordinates and find the unite vector along $\vec{A}$.
3. Find the distance between $(5, 3\pi/2, 0)$, and $(5, \pi/2, 10)$ in cylindrical coordinates, where: $\varphi = \frac{\pi}{2}$
4. Show that the two vectors: $\vec{A} = 4\hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $\vec{B} = \hat{\imath} + 4\hat{\jmath} - 4\hat{k}$ are perpendicular.
5. Given; $\vec{A} = 2\hat{\imath} + 4\hat{\jmath}$ and $\vec{B} = 6\hat{\imath} - 4\hat{k}$, find the smaller angle between them using: a. the cross product, b. the dot product.
6. Given; $\vec{F} = (y - 1)\hat{\imath} + 2x\hat{\jmath}$, find the vector at $(2,2,1)$ and its projection on B, where $\vec{B} = 5\hat{\imath} - \hat{\jmath} + 2\hat{k}$
7. Given; $\vec{A} = \hat{\imath} + \hat{\jmath}$ and $\vec{B} = \hat{\imath} + 2\hat{k}$ and $\vec{C} = 2\hat{\jmath} + \hat{k}$, a. find $(\vec{A} \times \vec{B}) \times \vec{C}$ and compare it with $\vec{A} \times (\vec{B} \times \vec{C})$, b. find $\vec{A}.\vec{B} \times \vec{C}$ and compare it with $\vec{A} \times \vec{B}.\vec{C}$.

8. Express the unite vector which is directed toward the origin from an arbitrary point on the plane $z = -5$, as shown in fig.



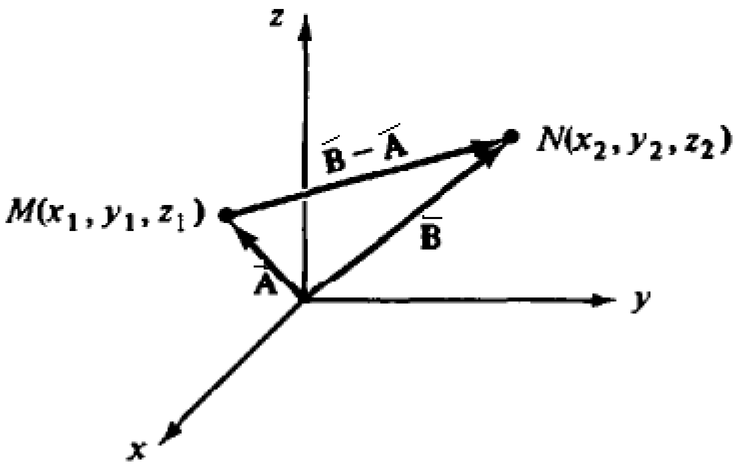
9. Using the appropriate differential elements, show that:
   a. the circumference of a circle of radius ρ_o is $2\pi\rho_o$.
   b. the surface area of a sphere of radius r_o is $4\pi r_o$
   c. the volume of a sphere of radius r_o is $(4/3)\ \pi r_o$
10. Obtain the expression for the volume of a sphere of a radius a from the differential volume.
11. Use the cylindrical coordinate system to find the area of the curved surface of a right circular cylinder, where: $\rho = 2m, h = 5m$ and $30 \leq \phi \leq 120$. See the following fig.



Solutions:

Q.1:

M and N are the two locations (in Cartesian coordinate) of those points in three dimensions. The two *position vectors* for them can be denoted by $\vec{A}$ and $\vec{B}$, respectively. Then:



$\vec{A} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ is the position vector from the origin to point to point N, and also:

$\vec{B} = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$ is the position vector from the origin to point M.

Then, the vector directed from M to N is given by the subtraction of $\vec{A}$ from $\vec{B}$; which gives:

$$\rightarrow \vec{B} - \vec{A} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$$

Q.2:

The position vector required in the question is:

$$\vec{A} = (0-2)\hat{\imath} + [-2-(-4)]\hat{\jmath} + (0-1)\hat{k} = -2\hat{\imath} + 2\hat{\jmath} - \hat{k}$$

$$|\vec{A}| = \sqrt{(-2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3$$

$$\rightarrow \hat{A} = \frac{\vec{A}}{|\vec{A}|} = -\frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} - \frac{1}{3}\hat{k}$$

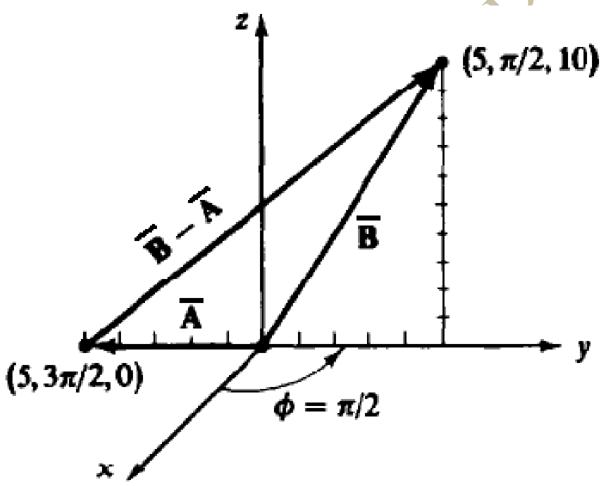
Q.3:

- Let: $M = (5,3\pi/2,0)$, and $N = (5,\pi/2,10)$
- The transformation equations from cylindrical to Cartesian coordinates are given as; $x = r\, cos\varphi$, $y = r\, sin\varphi$, $z = z$

For M: $= 5\cos\left(\frac{3\pi}{2}\right) = 0, y = 5\sin\left(\frac{3\pi}{2}\right) = -5, z = 0$, thus M will be (0,5,5) in cylindrical coordinate.

Also, for N: $= 5\cos\left(\frac{\pi}{2}\right) = 0, y = 5\sin\left(\frac{\pi}{2}\right) = 5, z = 10$, thus N will be $(0, 5, 10)$ in cylindrical coordinate.

- The position vector from the origin to M is: $\vec{A} = -5\hat{\jmath}$
- The position vector from the origin to N is: $\vec{B} = 5\hat{\jmath} + 10\hat{k}$



- The position vector from M to N in Cartesian coordinate then calculates as:

$\vec{B} - \vec{A} = 5\hat{\jmath} + 10\hat{k} - (-5\hat{\jmath}) = 10\hat{\jmath} + 10\hat{k},$

- Then, the required distance between the two points is;

$$\boxed{|\vec{A} - \vec{B}| = 10\sqrt{2}}$$

Q.4:

For the two vectors: $\vec{A} = 4\hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $\vec{B} = \hat{\imath} + 4\hat{\jmath} - 4\hat{k}$ we have:

$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|cos\theta = A_x.B_x + A_y\, B_y\, + A_z B_z$

Since the dot product contains $cos\theta$, a dot product of zero from any two nonzero vectors implies that the two vectors are perpendicular ($\theta = 90^o$), i.e:

$$|\vec{A}| = \sqrt{4^2 + 2^2 + (-1)^2} = \sqrt{21},\ |\vec{B}| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{33}$$

$$\therefore \vec{A}.\vec{B} = \sqrt{21}.\sqrt{33}cos90 = 0$$

Q.5:

(a) $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 4 & 0 \\ 0 & 6 & -4 \end{vmatrix} = -16\hat{\imath} + 8\hat{\jmath} + 12\hat{k}$

$|\vec{A}| = \sqrt{(2)^2 + (4)^2 + (0)^2} = 4.47$

$|\vec{B}| = \sqrt{(0)^2 + (6)^2 + (4)^2} = 7.21$

$|\vec{A} \times \vec{B}| = \sqrt{(-16)^2 + (8)^2 + (12)^2} = 21.54$, then;

since: $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|sin\theta$

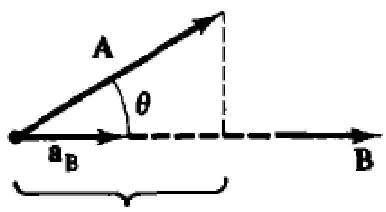
$sin\theta = \frac{21.54}{(4.47)(7.21)} = 0.668$ or $\theta = 41.9^o$

(b) $\vec{A}.\vec{B} = (2)(0) + (4)(6) + (0)(-4) = 24$

$cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|} = \frac{24}{(4.47)(7.21)} = 0.745$ or $\theta = 41.9^o$

Q.6:

$$F(2,2,1) = (2-1)\hat{\imath} + (2)(2)\hat{\jmath}$$
$$= \hat{\imath} + 4\hat{\jmath}$$



As indicated in fig., the projection of one vector on a second vector is obtained by expressing the unite vector in the direction of the second vector and taking the dot product.

$$Proj.\vec{A}\ on\ \vec{B} = \vec{A}.a_B = \frac{\vec{A}.\vec{B}}{|\vec{B}|}$$

Thus, at (2,2,1)

$$Proj.F\ on\ \vec{B} = \frac{\vec{F}.\vec{B}}{|\vec{B}|} = \frac{(1)(5)+(4)(-1)+(0)(2)}{\sqrt{30}} = \frac{1}{\sqrt{30}}$$

Q.7:

(a)

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2\hat{\imath} - 2\hat{\jmath} - \hat{k}$$

Then;

$$(\vec{A}\times\vec{B})\times\vec{C} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -2 & -1 \\ 0 & 2 & 1 \end{vmatrix} = -2\hat{\imath} + 4\hat{\jmath}$$

A similar calculation gives $(\vec{A}\times\vec{B})\times\vec{C} = 2\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$. Thus the parentheses that indicate which cross product is to be taken first are essential in the vector triple product.

(b)

$\vec{B}\times\vec{C} = -4\hat{\imath} - \hat{\jmath} + 2\hat{k}$. Then;

$\vec{A}.\vec{B}\times\vec{C} = (1)(-4) + (1)(-1) + (0)(2) = -5$

We have: $\vec{A}\times\vec{B} = 2\hat{\imath} - 2\hat{\jmath} - \hat{k}$. Then;

$\vec{A}\times\vec{B}.\vec{C} = (2)(0) + (-2)(2) + (-1)(1) = -5$

Parentheses are not needed in the scalar triple product since it has meaning only when the cross product is taken first. In general, it can be shown that:

$$\vec{A}.\vec{B}\times\vec{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

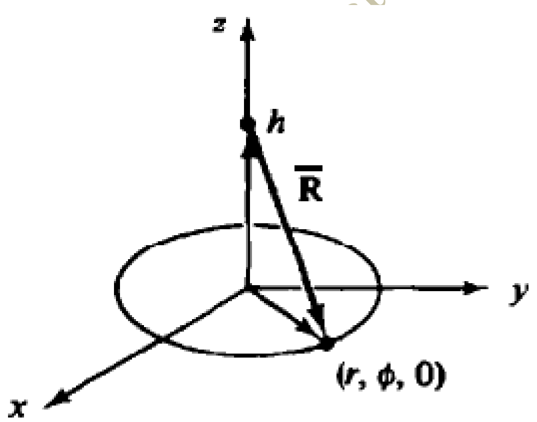
As long as the vectors appear in the same cyclic order the result is the same. The scalar triple products out of this cyclic order have a change in sign.

Q.8:

In order to calculate any unite vector, we have to know its vector and the vector magnitude. In this question, the position vector directed from point $(z = h)$ to some point $(r, \varphi, 0)$ in a cylinder base. The vector is directing from the point $(z = h)$ to $(r, \varphi, 0)$. In order to be able subtract the last point from the earlier one, we have to do the transformation from cylindrical to Cartesian coordinates.

$x = r\ cos\varphi\ , y = r\ sin\varphi\ , z = z \rightarrow$

$\therefore x = rcos\varphi, y = rsin\varphi, z = 0 \rightarrow (rcos\varphi, rsin\varphi, 0)$



$$\vec{R} = (rcos\varphi, rsin\varphi, 0) - h$$

$$\rightarrow \vec{R} = rcos\varphi\hat{r} + rsin\varphi\hat{\varphi} - h\hat{z}$$

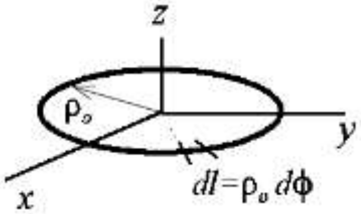
$$|\vec{R}| = \sqrt{(rcos\varphi)^2 + (rsin\varphi)^2 + h^2} = \sqrt{r^2(cos\varphi^2 + sin\varphi^2) + h^2}$$

$$= \sqrt{r^2 + h^2}$$

$$\boxed{\hat{R} = \frac{\vec{R}}{|\vec{R}|} = \frac{rcos\varphi\hat{r} + rsin\varphi\hat{\varphi} - h\hat{z}}{\sqrt{r^2 + h^2}}}$$

Q.9:

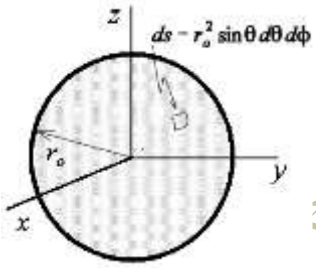
(a)



$$L = \int dl = \int_0^{2\pi} \rho_o d\phi = \rho_o \int_0^{2\pi} d\phi = \rho_o [\phi]_0^{2\pi}$$

$$\boldsymbol{L = 2\pi\rho_o}$$

(b)



$$S = \iint ds = \int_{\phi=0} \int_{\theta=0} r_o^2 sin\theta d\theta d\phi$$

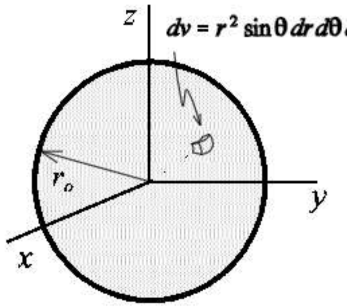
$$= r_o^2 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} sin\theta d\theta d\phi$$

$$= r_o^2 \int_{\phi=0}^{\pi} sin\theta d\theta \int_{\theta=0}^{2\pi} d\phi$$

$$= r_o^2 [-cos\theta]\text{I}_0^{\pi} [\phi]\text{I}_0^{2\pi}$$

$$= r_o^2 (2)(2\pi)$$

$$\boldsymbol{S = 4\pi r_o^2}$$



$$V = \iiint dv$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{r_o} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{r_o} r^2 dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \left[\frac{r^3}{3}\right]\Bigg|_0^{r_o} [-\cos\theta]\Big|_0^{\pi} [\phi]\Big|_0^{2\pi}$$

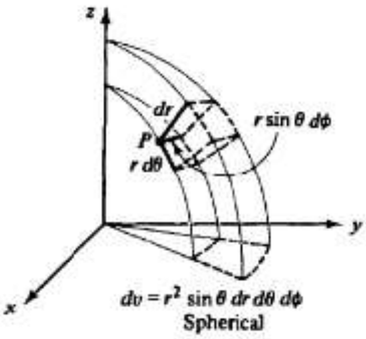
(c)

$$= \left(\frac{r_o^3}{3}\right)(2)(2\pi)$$

$$\boxed{V = \frac{4}{3}\pi r_o^3}$$

Q.10:

From the fig.;



$$dV = r^2 sin\theta dr d\theta d\phi$$

$$\boxed{V = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 sin\theta dr d\theta d\phi = \frac{4}{3}\pi a^3}$$

Q.11:

From the fig.; $\rho = 2m, z = 5m$ and $\phi = 30^o to 120^o$. Then; the differential surface element is: $ds = \rho d\phi dz$

$$= \int_0^5 \int_{\pi/6}^{2\pi/3} 2 d\phi\, dz$$

$$= 5\pi\, m^2$$

Exercises:

Problems

1–1. The vectors from the origin to the points A, B, C, D are

$$\begin{aligned} \mathbf{A} &= \mathbf{i} + \mathbf{j} + \mathbf{k}, \\ \mathbf{B} &= 2\mathbf{i} + 3\mathbf{j}, \\ \mathbf{C} &= 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}, \\ \mathbf{D} &= \mathbf{k} - \mathbf{j}. \end{aligned}$$

Show that the lines $\overline{AB}$ and $\overline{CD}$ are parallel and find the ratio of their lengths.

1–2. Show that the following vectors are perpendicular:

$$\begin{aligned} \mathbf{A} &= \mathbf{i} + 4\mathbf{j} + 3\mathbf{k}, \\ \mathbf{B} &= 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}. \end{aligned}$$

1–3. Show that the vectors

$$\begin{aligned} \mathbf{A} &= 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \\ \mathbf{B} &= \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}, \\ \mathbf{C} &= 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} \end{aligned}$$

form the sides of a right triangle.

1–4. By squaring both sides of the equation

$$\mathbf{A} = \mathbf{B} - \mathbf{C}$$

and interpreting the result geometrically, prove the "law of cosines."

1–5. Show that

$$\begin{aligned} \mathbf{A} &= \mathbf{i}\cos\alpha + \mathbf{j}\sin\alpha, \\ \mathbf{B} &= \mathbf{i}\cos\beta + \mathbf{j}\sin\beta \end{aligned}$$

are unit vectors in the xy-plane making angles α, β with the x-axis. By means of a scalar product, obtain the formula for $\cos(\alpha - \beta)$.