Chapter 5 : Sequences & Series

1- Sequences

Def. The sequence is an function where its domain the positive integer numbers(nature), and the co-domain is the complex numbers , denoted by $\{z_n\}$.

$\{z_n\}$=$\{z_1,z_2,z_3,\dots\dots,z_n,\dots\}$ $z_1 = f(1), z_2 = f(2), \dots\dots, z_n = f(n)$, where $n = 0,1,2, \dots$.

Ex. $\{i, -1, -i, 1, i, -1, \dots\dots\} = \{i^n\}$.

$\therefore \{z_n\} = i^n$.

Q. When $\{z_n\}$ become converges ?

Ans. If $\lim_{n\to\infty} z_n = z$, *then it called converges* , or by the definition of the sequence converges : $\forall \in > 0,\ \exists\, N > 0\ \ s.t.\ \ |z_n - z| < \in ,\ \ \forall\, n > N$.

Theorem.

Let $z_n = x_n + iy_n$, $(n = 1,2,3, \dots.)$, *and* $z = x + iy$, then

$$\lim_{n\to\infty} z_n = z \ \ iff \ \ \lim_{n\to\infty} x_n = x \ \ and \ \ \lim_{n\to\infty} y_n = y \ .$$

Ex. $\{\frac{i}{n}\}$? $\lim_{n\to\infty} \frac{i}{n} = \frac{i}{\infty} = 0$, the sequence is converges .

$\{(2i)^n\}$ is diverge.

2- Series

Def. let $\{z_n\}$ be sequence , then the sum $\sum_{n=1}^{\infty} z_n = z_1,z_2,z_3,\dots\dots,z_n,\dots$

$S_1 = z_1$

$S_2 = z_1 + z_2$

$S_3 = z_1 + z_{2+} z_3$

-- -- --

$S_n = z_1 + z_2 + z_3 + \cdots \dots.. + z_n$. S_n *is called the Partial Sum Series* $\{S_n\}$ and

$\lim_{n\to\infty} S_n = S = \sum_{n=1}^{\infty} z_n$.

Power Series

$\sum_{n=0}^{\infty} a_n.(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \ldots a_n(z-z_0)^n + \cdots \ldots$,

Where a_n *and* z_0 are complex numbers.

1- If $z = z_0 \Rightarrow \sum_{n=0}^{\infty} a_n.(z-z_0)^n = a_0$.
2- If $z_0 = 0 \Rightarrow \sum_{n=0}^{\infty} a_n.(z-z_0)^n = \sum_{n=0}^{\infty} a_n.z^n$.

Cauchy ' test

$\lim_{n\to\infty} |a_n(z-z_0)^n|^{\frac{1}{n}} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} |z-z_0| = \frac{1}{R}|z-z_0|$. draw

$\therefore \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R}$, R is the radius of convergence .

$$If \ \frac{1}{R}|z-z_0| < 1 \Rightarrow |z-z_0| < R \ --converges,$$

$$|z-z_0| > R-- \text{diverges} .$$

Ex. $f(z) = e^z, sinz \ and \ cosz \ are \ converges$.

Ex. . Discuss the convergence of $\sum_{n=0}^{\infty} 3^n (z-i)^n$?

$\frac{1}{R} = |a_n|^{\frac{1}{n}} = |3^n|^{\frac{1}{n}} = 3 \Rightarrow R = \frac{1}{3}$

$\therefore |z-i| < \frac{1}{3} \ --converges,$

$|z-i| > \frac{1}{3} \ --diverges$.

Ex. Discuss the convergence of $\sum_{n=1}^{\infty} (\frac{n+1}{n})^{n^2} (z-1)$?

$a_n = (\frac{n+1}{n})^{n^2}$, $z = 1$,

$\lim_{n\to\infty} |(\frac{n+1}{n})^{n^2}| = \lim_{n\to\infty} (\frac{n+1}{n})^n = \lim_{n\to\infty} (1+\frac{1}{n})^n = e$,

$\therefore |z-1| < e \ \ is \ the \ circle \ of \ convergence.$

$\therefore \sum_{n=1}^{\infty} (\frac{n+1}{n})^{n^2} (z-1) \ --converges.$

Taylor Series

Theorem.

Let f be analytic everywhere inside the circle C_0 with center z_0 and radius r_0,

then at each point z inside C_0 :

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + f''(z_0)\frac{(z - z_0)^2}{2!} + f'''(z_0)\frac{(z - z_0)^3}{3!} + \cdots$$

$\ldots + f^{(n)}(z_0)\frac{(z-z_0)^n}{n!} + \cdots \ldots$ draw

Proof. Let $|z - z_0| = r \quad , \quad r < r_0 \;\; and \;\; let \;\; s \in C_1,$

$\because \;\; f(z) \; is \; analytic$

$\therefore f(z) = \frac{1}{2i\pi}\int_{C_1}\frac{f(s)}{s-z}\,ds$

$\frac{1}{s-z} = \frac{1}{(s-z)-(z-z_0)} = \frac{1}{s-z_0}\left(\frac{1}{1-\frac{z-z_0}{s-z_0}}\right)$

If $\frac{z-z_0}{s-z_0} < 1$, we can write it by the geometric series :

$\therefore \frac{1}{s-z} = \frac{1}{s-z_0}[1 + \frac{z-z_0}{s-z_0} + (\frac{z-z_0}{s-z_0})^2 + \cdots + (\frac{z-z_0}{s-z_0})^{n-1} + (\frac{z-z_0}{s-z_0})^n.\left(\frac{1}{1-\frac{z-z_0}{s-z_0}}\right)]$

$\quad = \frac{1}{s-z_0} + (z - z_0)\frac{1}{(s-z_0)} + (z - z_0)^2\frac{1}{(S-z_0)^2} + (z - z_0)^{n-1}\frac{1}{(S-z_0)^n} + (\frac{z-z_0}{s-z_0})^n\frac{1}{(S-z)}$

$\therefore f(z) = \frac{1}{2i\pi}\int_{C_1}\frac{f(s)ds}{s-z_0} + \frac{z-z_0}{2i\pi}\int_{C_1}\frac{f(s)ds}{(s-z_0)^2} + \frac{(z-z_0)^2}{2i\pi}\int_{C_1}\frac{f(s)ds}{(s-z_0)^3} + \cdots + \frac{(z-z_0)^{n-1}}{2i\pi}\int_{C_1}\frac{f(s)ds}{(s-z_0)^n} + U_n$,

Where $U_n = \frac{1}{2i\pi}\int_{C_1}(\frac{z-z_0}{s-z_0})^n\frac{f(s)}{s-z}\,ds$,

By using C.I.F. 2, we get

$$\therefore f(z) = f(z_0) + f'(z_0)(z - z_0) + f''(z_0)\frac{(z - z_0)^2}{2!} + \cdots + f^{(n-1)}(z_0)\frac{(z - z_0)^{n-1}}{(n-1)!} + U_n\,.$$

$$U_n = \frac{(z - z_0)^n}{2i\pi}\int_{C_1}\frac{f(s)}{(s - z_0)^{n+1}}.\frac{1}{1 - \frac{z - z_0}{s - z_0}}\,ds = \frac{(z - z_0)^n}{2i\pi}\int_{C_1}\frac{f(s)}{(s - z_0)^{n+1}}.\frac{s - z_0}{s - z}\,ds$$

$\leq \frac{|z-z_0|^n}{2\pi}\int_{C_1}\frac{|f(s)|}{|s-z_0|^n}.\frac{1}{|s-z|}|ds|$

$\leq \frac{r^n.M}{2\pi}\int_{C_1}\frac{|ds|}{|s-z|}$,

$|s - z| = |(s - z_0) - (z - z_0)| \geq |s - z_0| - |z - z_0| \geq r_1 - r$

$\therefore U_n \leq \frac{M}{2\pi}(\frac{r}{r_1})^n.\frac{1}{r_1-r}.2\pi r_1 \leq \frac{Mr_1}{2\pi}(\frac{r}{r_1})^n \to 0 \;\; when \;\; n \to \infty \quad , \; r_1 > r$

i.e.

$\lim_{n\to\infty} U_n = 0 \Rightarrow f(z) = f(z_0) + f'(z_0)(z - z_0) + f''(z_0)\frac{(z-z_0)^2}{2!} + \cdots + f^{(n)}(z_0)\frac{(z-z_0)^n}{n!}$. ■

If $z_0 = 0 \Rightarrow \boldsymbol{f(z) = f(0) + f'(0)z + f''(0)\frac{z^2}{2!} + \cdots + f^{(n)}(0)\frac{z^n}{n!}}$, this series

Is called **Maclourian's Series .**

Ex. Expand the function $f(z) = e^z$ in the Maclourian's Series .

$f(0) = e^0 = 1$, $f'(0) = 1$, $f''(0) = 1$,

$\therefore f(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \ldots = \sum_{n=o}^{\infty} \frac{z^n}{n!}$.

Q. Expand the following functions in the form of Taylor's series :

1- $sinz$, $z = 0$, 2- $cosz$, $z = 0$, 3- $\log(1 + z^2)$, $z = 1$.

Laurent's Series

Theorem.

Let $f(z)$ is analytic in a Ring-Shaped region enclosed by two concentric circles C_1 and C_2 with center z and radius r , r respectively , then

$$\boldsymbol{f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n(z - z_0)^{-n}}$$

Analytic Part principle Part

Also we can write the Laurent's series by $\boldsymbol{f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n}$,

Where $a_n = \frac{1}{2i\pi}\int_{C_1} \frac{f(s)}{z-s} ds \ldots I_1$, $b_n = \frac{1}{2i\pi}\int_{C_2} f(s)(s - z_0)^{n-1} ds \ldots I_2$

Proof.

$I_1 \ldots this\ integral\ is\ proved\ (Taylor's\ Series\ ).$

$I_2 = \frac{1}{2i\pi}\int_{C_2} \frac{f(s)}{z-s} ds$,

$$\frac{1}{z-s}=\frac{-1}{s-z}<|-1\left[\frac{1}{(s-z_0)-(z-z_0)}\right]|$$

Ex. Expand the function $f(z)=\frac{-1}{(z-1)(z-2}$ in the form of Laurent's series in

The ring $1<|z|<2$? draw

Sol. $f(z)=\frac{1}{z-1}-\frac{1}{z-2}$,

$\forall\ |z|>1$,

$\frac{1}{z-1}=\frac{1}{z\left(1-\frac{1}{z}\right)}=\frac{1}{z}\ [1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\frac{1}{z^n}+\cdots\ =\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots=\sum_{n=1}^{\infty}\frac{1}{z^n}$,

and $\forall\ |z|<2$,

$$-\frac{1}{z-2}=\frac{1}{2-z}=\frac{1}{2\left(1-\frac{z}{2}\right)}=\frac{1}{2}[1+\frac{z}{2}+(\frac{z}{2})^2+(\frac{z}{2})^3+\cdots++(\frac{z}{2})^n\ldots$$

$$=\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\cdots\ =\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

$\therefore\ f(z)=\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}+\sum_{n=1}^{\infty}\frac{1}{z^n}$.

Chapter 6 : Singularity Of Analytic function & Residues

Def. The point z_0 is called a **singular point** of the function $w = f(z)$, if $f(z)$ is not analytic at z_0 but it is analytic in the neighborhood of z_0 .

1- Non-Isolated Singularity

Def. The point z_0 is called a **Non-Isolated** singular point for f(z) , if z_0 is a singular point for f(z) and the neighborhood of z_0 contains at least another singular point different at z_0 .

Ex. $f(z) = \frac{1}{sin\frac{\pi}{z}}$, $z = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3} \ldots$ are Non-Isolated singular points for f(z).

2- **Isolated Singularity**

Def. The point z_0 is called a **Isolated** singular point for $f(z)$,if in the neighborhood of z_0, the function $f(z)$ is analytic .

Ex. $f(z) = \frac{1}{z}$, $z = 0$ *is* Isolated singular point for $f(z)$.

Ex. $f(z) = \frac{2i}{z^2+1}$, $z = \pm i$ *are* Isolated singular points for $f(z)$.

Classification of Isolated Singularity

If the function $f(z)$ is analytic $\forall\, z$ which satisfy $0 < |z - z_0| < \rho$, then

$$\boldsymbol{f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n} \text{ , } (\textit{laurent's Series}).$$

Def. 1. If the principle part 0f $f(z)$ at $z = z_0$ contains no terms , the z_0 is said To be **Removable singularity** of $f(z)$.

Ex. $f(z) = \frac{sinz}{z}$, $z = 0$ is removable singular point, because $\frac{sinz}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$,

It is clear that the $\lim_{z\to 0}\frac{sinz}{z} = 1$.

Ex. $f(z) = \frac{z^3+8}{z+2}$, $z = -2$ is removable singular point.

Def. 2. If the principle part 0f $f(z)$ at $z = z_0$ consists of a finite number of terms (m), We say that z_0 is a **Pole** of order m of $f(z)$.

In case if z_0 is a pole of order $= 1$, it called a **simple pole** .

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m} . \quad b_m \neq 0$$

If z_0 is a pole of $f(z) \Rightarrow \lim_{z\to\infty} f(z) = \infty$.

Ex. $f(z) = \frac{1}{(z-2)^3}$, $z = 2$ *is pole of order* $m = 3$.

Ex. $f(z) = \frac{e^z-1}{z^2}$, $z = 0$ *is a simple pole* $(m = 1)$, *because*

$$\frac{e^z-1}{z^2} = \frac{1}{z^2}\left(\sum_{n=0}^{\infty}\frac{z^n}{n!} - 1\right) = \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \cdots .$$

Ex. $f(z) = \frac{sinhz}{z^3}$, $z = 0$ *is apol of order* $m = 2$, *because* $\frac{sinhz}{z^3} = \frac{1}{z^2} + \frac{1}{3!} + \frac{z^2}{5!} + \cdots$.

Def. 3. If the principle part 0f $f(z)$ at $z = z_0$ contains infinite number of terms , then z_0 is called an isolated **Essential** singularity of $f(z)$.

Ex. $f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$, $z = 0$ *is essential singular point*.

Ex. $f(z) = (z - 3)\sin\left(\frac{1}{z+2}\right)$, $z = -2$ *is essential singular point*.

Proof. Let $u = z + 2 \Rightarrow z - 3 = u - 5$,

$$\therefore f(z) = (u-5)sin\frac{1}{u} = (u-5)\left[\frac{1}{u} - \frac{1}{3!u^3} + \frac{1}{5!u^5} - \cdots\right] = 1 - \frac{5}{u} - \frac{1}{3!u^2} + \frac{5}{3!u^3} + \cdots,$$

$$= 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3} + \frac{1}{120(z+2)^4} - \cdots \quad \blacksquare$$

Residues

Def. Let $f(z)$ is analytic function , z_0 is isolated singular point of $f(z)$, we can find a neighborhood of z_0 : $0 < |z - z_0| < r$ such that the $f(z)$ is analytic except at z_0.

Then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{a_{-1}}{(z-z_0)} + \frac{a_{-2}}{(z-z_0)^2} + \cdots$,

where $a_n = \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}} dz$, $n = 0, \pm 1, \pm 2, \ldots$

In case when $n = -1 \Rightarrow a_{-1} = \frac{1}{2\pi i}\int_C f(z)dz$, *where C is a simple closed curve ,*

The $\boldsymbol{a_{-1}}$ is called the **<u>Residue</u>** of $f(z)$ at z_0 and written by $\boldsymbol{Res(f(z), z_0) = a_{-1}}$.

$$\therefore \int_C \boldsymbol{f(z)dz} = \boldsymbol{2\pi i. Res(f(z), z_0)} .$$

Ex. Find the residue of the function $f(z) = e^{-\frac{1}{z}}$?

$$\because e^{-\frac{1}{z}} = 1 - \frac{1}{z} + \frac{1}{2!\,z^2} - \frac{1}{3!\,z^3} + \cdots$$

$$\therefore Res\left(e^{-\frac{1}{z}}, 0\right) = -1 \ .$$

Ex. Compute the integral $\int_C \frac{e^z}{z^2}\,dz$, *where* $C: |z| = 1$?

$$f(z) = \frac{e^z}{z^2} = \frac{1}{z^2}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)$$

$$= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \cdots \qquad ,$$

$$\therefore Res\left(\frac{e^z}{z^2}, 0\right) = 1 \quad \Rightarrow \int_C \frac{e^z}{z^2}\,dz = 2\pi i \,.\, 1 = 2\pi i \quad .$$

Residues Theorem

Let $f(z)$ is analytic function on & in the simple closed curve C except at a finite number Of points z_k , where z_k are isolated singular points , then

$$\int_C \boldsymbol{f(z)dz} = \boldsymbol{2\pi i \sum_{k=1}^{n} Res(f, z_k)} \ .$$

notes.

1- If z_k are removable singular points of $f(z)$, then $\mathrm{Res}(\mathrm{f}, \mathrm{z_k}) = 0 \ \forall\, k = 1,2,3, \dots, n$, because the Laurent series become the Taylor series $\Rightarrow a_{-1} = 0$.

2- If z_k are essential singular points of $f(z)$, in this case we expand the $f(z)$ in the Laurent series and calculate the residues of $f(z)$ at every point z_k, $k = 1,2,3, \dots, n$.

3- If z_k are a poles of $f(z)$, we find the order of the pole, then we calculate the residue of $f(z)$, if it is difficult to do that , so we use the following theorem :

Calculation of Residues Theorem

Let $f(z)$ has a pole of order n *at* z_0. Then

$$\boldsymbol{Res(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{(n-1)}}{dz^{(n-1)}} [(z - z_0)^n f(z)]}.$$

Proof.

$$\because z_0 \text{ is a pole of order } n \Rightarrow f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k \ , a_{-n} \neq 0,$$

$$(z-z_0)^n f(z) = \sum_{k=-n}^{\infty} a_k (z-z_0)^{n+k}$$

$$= a_{-n} + a_{-n+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{n-1} + \sum_{k=0}^{\infty} a_k (z-z_0)^{n+k}$$

It is clear that this series is the Taylor series and we can find the derivative for all terms, Now we find the derivative of order $(n-1)$:

$\frac{d^{(n-1)}}{dz^{(n-1)}}[(z-z_0)^n . f(z)] = (n-1)!\,.a_{-1} + \sum_{k=0}^{\infty} \frac{d^{(n-1)}}{dz^{(n-1)}} a_k (z-z_0)^{n+k}$

When $z \to z_0$, $the\ right\ hand\ side\ \to constant$

$\therefore \lim_{z\to z_0} \frac{d^{(n-1)}}{dz^{(n-1)}}[(z-z_0)^n . f(z)] = (n-1)!\,.a_{-1}$

$\therefore a_{-1} = \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{(n-1)}}{dz^{(n-1)}}[(z-z_0)^n . f(z)]$ ■

<u>Note.</u> When $n = 1 \Rightarrow \boldsymbol{Res(f, z_0) = \lim_{z\to z_0}[(z-z_0).f(z)}$.

Ex. Using the theorem of calculation of residues to find $\int_C \frac{dz}{(z-1)(z+1)}$, $C: |z| = 3$?

Sol. $z = \pm 1$ are simple poles inside the circle, now by using the theorem above

$$Res(f,1) = \lim_{z\to 1} [(z-1).\frac{1}{(z-1)(z+1)}] = \lim_{z\to 1} \frac{1}{(z+1)} = \frac{1}{2}$$

$$Res(f,-1) = \lim_{z\to -1} [(z+1).\frac{1}{(z-1)(z+1)}] = \lim_{z\to -1} \frac{1}{(z-1)} = -\frac{1}{2}$$

$$\therefore \int_C \frac{dz}{(z-1)(z+1)} = 2\pi i.[Res(f,1) + Res(f,-1)] = 2\pi i\left[\frac{1}{2} + \left(-\frac{1}{2}\right)\right] = 0.$$

Ex. Compute $\int_C \frac{e^{iz}+sinz}{(z-\pi)^3}\, dz$, $C: |z-3| = 1$?

Sol. $z = \pi$ is a pole of order $n = 3$, by the theorem

$$Res(f,\pi) = \frac{1}{(3-1)!} \lim_{z\to\pi} \frac{d^{(3-1)}}{dz^{(3-1)}}\left[\left(z-\pi)^3 . \frac{e^{iz}+sinz}{(z-\pi)^3}\right] = \frac{1}{2}\right.$$

$$\therefore \int_C \frac{e^{iz}+sinz}{(z-\pi)^3}\, dz = 2\pi i.\frac{1}{2} = \pi i \ .$$