Definition:- Let $X \neq \emptyset$, The distance function (metric) is a function $d: X \times X \rightarrow \mathbb{R}$ satisfy the following conditions:-

1.$d(x,y) \geq 0 \quad \forall x,y \in X$.

2.$d(x,y)=0$ if and only if $x=y$.

3.$d(x,y)=d(y,x) \quad \forall x,y \in X$.

4.$d(x,z) \leq d(x,y)+d(y,z) \quad \forall x,y,z \in \mathbb{Z}$

And the order pair (x, d) is called metric space.

Example:- Anon –empty set X of complex number is metric space with $d(\lambda,\mu)=|\lambda-\mu|$

Sol:- $\lambda=a_1+ib_1$, $\mu=a_2+ib_2$

$d: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$, since $|\lambda-\mu| = |a_1+ib_1-a_2+ib_2| = \sqrt{(a1-a2)^2+(b1-b2)^2} \geq 0$

1. $d(\lambda, \mu)=|\lambda-\mu| \geq 0 \quad \forall \lambda, \mu \geq 0$

2. if $d(\lambda, \mu)=0 \rightarrow |\lambda-\mu|=0 \rightarrow \lambda-\mu=0 \rightarrow \lambda=\mu$

3. $d(\lambda, \mu)=|\lambda-\mu|$

$= |(-1)(\mu-\lambda)| = |-1| \, |\mu-\lambda| = |\mu-\lambda| = d(\mu, \lambda)$

4. $d(\lambda, \mu)=|\lambda-\mu|=|\lambda- z+ z-\mu| \leq |\lambda-z| + |z-\mu| \leq d(\lambda, z)+d(z, \mu)$

$\therefore$ d is metric and $(\mathbb{C},d)$ is metric space called usual m.s in $\mathbb{C}$

Example:- Let X be any non-empty set, Define

$$d(x,y)=\begin{cases} 1 & if\ x \neq y \\ 0 & if\ x = y \end{cases}$$

then d is metric (called distanic metric) or discrete metric space.

Sol:-

1- $d(x,y)=1>0$ if $x\neq y$.
2- if $x=y \rightarrow d(x,y)=0$.
3- If $x\neq y \rightarrow d(x,y)=1=d(y,x)$

If $x=y \rightarrow d(x,y)=0=d(y,x)$.

4- Let $x,y,z \in X$.

a. If $x=y=z$

$$d(x,y) \leq d(x,z)+d(z,y)$$

$$0 \leq 0 + 0$$

b. If $x=y\neq z$

$$d(x,y) \leq d(x,z)+d(z,y)$$

$$0 \leq 1 + 1$$

c. If $x\neq y=z$

$$d(x,y) \leq d(x,z)+d(z,y)$$

$$1 \leq 1 + 0$$

d. If $x\neq y\neq z$

$$d(x,y) \leq d(x,z)+d(z,y)$$

$$1 \leq 1 + 1$$

$\therefore (x,d)$ is discrete metric space

Remark:-

Every non- empty subset of metric space is metric space.

i.e if (x,d) metric space and $N\neq \emptyset \ni N \subseteq x$ then (N,d) is metric space

Examples:-

1) Let $X=\mathbb{R}$,d: $X\times X \rightarrow \mathbb{R}$ be a function defined by $d(x,y)= |x,y|$, Then (x,d) is a metric space (called usual metric space).

2) Let X=C[a,b] (the set of all continuous function defined on closed interval[a,b]) for f, g $\in X$ $d(f,g)=\max|f(x)-g(x)|$, $x\in[a,b]$ then (x,d) is a m.s.
3) Let $X=\mathbb{R}^2$, $X=(x_1,y_1)$, $Y=(x_2,y_2)$ Take:-
   $d1(x,y)=\sqrt{(x2-x1)^2+(y2-y1)^2}$
   $d2(x,y)=|x_1-x_2|+|y_1-y_2|$
   $d3(x,y)=\max(|x_1-x_2|,|y_1-y_2|)$
   $\therefore$ d1,d2 and d3 are metric.

Definition: A sequence x_n in a metric space is said to be convergent to point X in case $d(x_n, x)\to 0$ as $n\to\infty$
This means given any number $\varepsilon>0$ $\exists$ positive integer N such that $d(x_n,x)<\varepsilon$ $\forall n\geq N$ otherwise is divergent.
Remark:- $X_n\to x$ as $n\to\infty \sim \lim_{n\to\infty} xn = x$.
Definition:- A sequence Xn in a metric space is said to be Cauchy in case $d(x_m,x_n)\to 0$ as $m,n\to\infty$ This means: given any $\varepsilon>0$, $\exists$ positive integer N$\ni$ $d(x_m,x_n)<\varepsilon$ $\forall n,m\geq N$.

Remark:- Every convergent sequence is Cauchy sequence but the converse is not necessary to be true.
Let $\{X_n\}$ is a convergent sequence at a point x_o T.P$\{X_n\}$is Cauchy sequence.
Since $\{X_n\}$ is convergent at a point x_o.
$\forall\varepsilon>0$ $\exists$ positive integer N$\ni$ $d(x_n,x_o)\leq\frac{\varepsilon}{2}$ $\forall n\geq N$ for any $n\geq N$, $m\geq N$

$d(x_n,x_m)\leq d(x_n,x_o)+d(x_m,x_o)$
$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

$\therefore d(x_n,x_m)<\varepsilon$
$\therefore\{X_n\}$ is Cauchy sequence but consider the example.
(R-{0},d) be a usual m.s
The sequence $\{\frac{1}{n}\}$ is Cauchy in m.s (R-{0},d) but not convergent because $\{\frac{1}{n}\}$ convergent to 0 and $0\notin R-\{0\}$.

Definition:- A metric space is said to be complete if every Cauchy sequence is convergent.

Example:- Every discrete metric space is complete.

Sol:- $d(x,y)=\begin{cases} 0 & x=y \\ 1 & x\neq y \end{cases}$

Let X_n be Cauchy sequence

$\rightarrow d(x_n,x_m)\rightarrow 0$ as $n,m\rightarrow\infty$

$\rightarrow x_n=x_m$

$\rightarrow < Xn > = < x_1,x_2,\ldots,x_n, x,x,\ldots\ldots >$

$\rightarrow d(x_n,x)\rightarrow 0$ as $n\rightarrow\infty$

$\therefore X_n\rightarrow x$

$\therefore X_n$ convergent

$\therefore$ every discrete m.s is complete.

Example:- Let X be a space of all complex sequence $\{X_i\}$ and d be a metric defined on X as follows:

$d(x,y)=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|Xi-Yi|}{1+|Xi-Yi|}$

Where $X=(x_1,x_2,\ldots,x_n,\ldots)$

$Y=(y_1,y_2,\ldots,y_n,\ldots)$

1) $d(x,y)=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|Xi-Yi|}{1+|Xi-Yi|}\geq 0$

$\rightarrow d(x,y)\geq 0 \;\forall\, x,y\in X.$

2) $d(x,y)=0\rightarrow\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|Xi-Yi|}{1+|Xi-Yi|}=0\rightarrow X_i=Y_i \quad\forall i$

$X=Y\rightarrow X_i=Y_i \quad\forall i\rightarrow\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|Xi-Yi|}{1+|Xi-Yi|}=0$

$\rightarrow d(x,y)=0.$

3) $d(x,y)=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|Xi-Yi|}{1+|Xi-Yi|}=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|Yi-Xi|}{1+|Yi-Xi|}$

$=d(y,x).$

4) Let x,y,z $\in$X Where Z=(z_1,z_2,….,z_n,…)

$$d(x,y)= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|Xi-Yi|}{1+|Xi-Yi|} =\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|Xi-Zi+Zi-Yi|}{1+|Xi-Zi+Zi-Yi|}$$

$$\leq \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|Xi-Zi|}{1+|Xi-Zi|} + \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|Zi-Yi|}{1+|Zi-Yi|}$$

Example:

Suppose that P$\in$N consider the set ℓ^P of all infinite Sequence X=(x_1,x_2,…..) of complex number such that the series $\sum|Xi|^P<\infty$ is converges we can defined a metric ℓ^P as :

$$d(x,y)= (\sum_{i=1}^{n} |Xi-Yi|^P)^{\frac{1}{P}} .$$

Sol:-

1) $|X_i-Y_i| > 0 \rightarrow (\sum_{i=1}^{n} |Xi-Yi|^P)^{\frac{1}{P}} >0 \rightarrow d(x,y) >0.$

$d(x,y)=0 \rightarrow (\sum_{i=1}^{n} |Xi-Yi|^P)^{\frac{1}{P}} =0$

$\Leftrightarrow |Xi-Yi|^P=0 \Leftrightarrow |Xi-Yi| = 0 \Leftrightarrow X_i=Y_i$.

2) $d(x,y)= (\sum_{i=1}^{n} |Xi-Yi|^P)^{\frac{1}{P}} = \sum_{i=1}^{n}(|-1| |Yi-Xi|^P)^{\frac{1}{P}}$

$= (\sum_{i=1}^{n} |Yi-Xi|^P)^{\frac{1}{P}} = d(y,x).$

3) $d(x,z)= (\sum_{i=1}^{n} |Xi-Zi|^P)^{\frac{1}{P}} = (\sum_{i=1}^{n} |Xi-Yi+Yi-Zi|^P)^{\frac{1}{P}}$

$\leq (\sum_{i=1}^{n} |Xi-Yi|^P + |Yi-Zi|^P)^{\frac{1}{P}}$

$\leq$ d(x,y)+d(y,z).

Example:-

Consider the set of all bounded infinite sequence of complex number ℓ^∞ , for any sequence $\in \ell^\infty$, X=(x_1,x_2,…..) and Y=(y_1,y_2,…) Write d(x,y)=Sup $|X_i-Y_i|$, i$\in$N

Then (ℓ^∞,d) is a metric space on ℓ^∞.

Sol:-

1) Let $X,Y \in \ell^{\infty} \rightarrow X=(x_1,x_2,\dots)$, $Y=(y_1,y_2,\dots.)$

$|X_i-Y_i|>0 \rightarrow \text{Sup } |X_i-Y_i|>0$

$\therefore d(x,y)>0$

$d(x,y)=0 \rightarrow \text{Sup } |X_i-Y_i|=0$

$\rightarrow |X_i-Y_i|=0 \rightarrow X_i-Y_i=0 \rightarrow X_i=Y_i$

Let $X_i=Y_i \rightarrow X_i-Y_i=0$

$\rightarrow |X_i-Y_i|=0 \rightarrow \text{Sup } |X_i-Y_i|=0 \rightarrow d(x,y)=0.$

2) $d(x,y)=\text{Sup } |Xi-Yi| = \text{Sup } |Yi-Xi| = d(y,x).$

3) Let $Z \in \ell^{\infty} \rightarrow Z=(z_1,z_2,\dots.)$

Then $d(x,z)=\text{Sup } |X_i-Z_i|$

$\leq \text{Sup } \big| |X_i-Y_i| + |Y_i-Z_i| \big|$

$\leq \text{Sup } |X_i-Y_i| + \text{Sup } |Y_i-Z_i|$

$\leq d(x,y) + d(y,z)$

$\therefore (\ell^{\infty}, d)$ is m.s .

Exe:

1)Consider the set ℓ^2 of all infinite sequence $X=(x_1,x_2,\dots.)$ of complex number such that $\sum |Xi|^2 < \infty$ is convergent , for any two sequence $X=(x_1,x_2,\dots)$ and $Y=(y_1,y_2,\dots)$ Write $d(x,y)=\left(\sum_{i=1}^{n} |Xi-Yi|^2\right)^{\frac{1}{2}}$ the check (ℓ^2,d) is a m.s .

2) Consider the set C[a,b] of all bounded complex valued function defined on an interval [a,b] forever $f,g \in C[a,b]$ with

$d(f,g)=\text{Sup } |f(t)-g(t)|, \quad t \in [a,b].$