Chapter three Mathematics *

Modular Arithmetic

several important cryptosystems make use of modular arithmetic. This is when the answer to a calculation is always in the range *0 – m* where m is the **modulus.**

(a mod n) means the remainder when a is divided by n.

a mod n = r

a div n=q

a = qn + r

r = a – q * n

Example :- if a=13 and n=5, find q and r.

q=13 div 5=2 and r=13-2 *5=**3** which is equivalent to (13 mod 5)

Example :- find (-13 mod 5).

This can be found by find the number (b) where 5*b >13 then let b=3 and 5*3=15 which is less than 13 so

-13 mod 5=5*3-13=**2**

Properties of Congruences. *

Two numbers ***a*** and ***b*** are said to be "*congruent modulo **n***" if *

$$(a \bmod n) = (b \bmod n) \rightarrow a \equiv b \pmod{n}$$

The difference between ***a*** and ***b*** will be a multiple of ***n*** So ***a-b*** = ***kn*** for some value of ***k*** *

Examples $4 \equiv 9 \equiv 14 \equiv 19 \equiv -1 \equiv -6 \bmod 5$ *

73 ≡ 4(mod 23

Properties of Modular Arithmetic. *

١. $[(a \bmod n) + (b \bmod n)] \bmod n = (a + b) \bmod n$

٢. $[(a \bmod n) - (b \bmod n)] \bmod n = (a - b) \bmod n$

٣. $[(a \bmod n) \times (b \bmod n)] \bmod n = (a \times b) \bmod n$

Examples *

$11 \bmod 8 = 3$; $15 \bmod 8 = 7$

$[(11 \bmod 8) + (15 \bmod 8)] \bmod 8 = 10 \bmod 8 = 2$

$(11 + 15) \bmod 8 = 26 \bmod 8 = 2$

$[(11 \bmod 8) - (15 \bmod 8)] \bmod 8 = -4 \bmod 8 = 4$

$(11 - 15) \bmod 8 = -4 \bmod 8 = 4$

$[(11 \bmod 8) \times (15 \bmod 8)] \bmod 8 = 21 \bmod 8 = 5$

$(11 \times 15) \bmod 8 = 165 \bmod 8 = 5$

- **Exponentiation** is done by repeated multiplication, as in ordinary arithmetic.
- Example

$$To\ find\ (11^7 \bmod 13)\ do\ the\ followings$$

$$11^2 = 121 \equiv 4 (\bmod 13)$$

$$11^4 (11^2)^2 \equiv 4^2 \equiv 3 (\bmod 13)$$

$$11^7 \equiv 11 \times 4 \times 3 \equiv 132 \equiv 2 (\bmod 13)$$

Greatest Common Divisor(GCD). *

* Let *a* and *b* be two non-zero integers. The greatest common divisor of *a* and *b*, denoted gcd(a,b) is the largest of all common divisors of *a* and *b*.

* When gcd(*a*,*b*) = 1, we say that *a* and *b* are *relatively prime*.

* It can be calculated using the following equation: -

$GCD(a,b)=GCD(b,a \bmod b)$

* Example :- find the GCD(72,48).

GCD(89,25)=GCD(25, 89 mod 25)= GCD(25, 14)

GCD(25, 14)=GCD(14, 25 mod 14)= GCD(14,11)

GCD(14,11)=GCD(11, 14 mod 11)= GCD(11,3)

GCD(11,3)=GCD(3, 11 mod 3)=GCD(3, 2)

GCD(3,2)=GCD(2, 3 mod 2)=GCD(2,1)

GCD(2,1)=GCD(1, 2 mod 1)=GCD(1,0) so the GCD(89,25)=1

Least Common Multiple (LCM).

- The least common multiple of the positive integers a and b is the smallest positive integer that is divisible by both a and b.
- The least common multiple of a and b is denoted by LCM(a, b).
- It can be calculated using the following equation: -

$$LCM(a, b)=|a * b| / GCD(a, b)$$

- Example :- find the LCM(354,144).

GCD(354,144)=GCD(144,354 mod 144)=GCD(144,66)

GCD(144,66)=GCD(66, 144 mod 66)= GCD(66,12)

GCD(66,12) =GCD(12, 66 mod 12)=GCD(12,6)

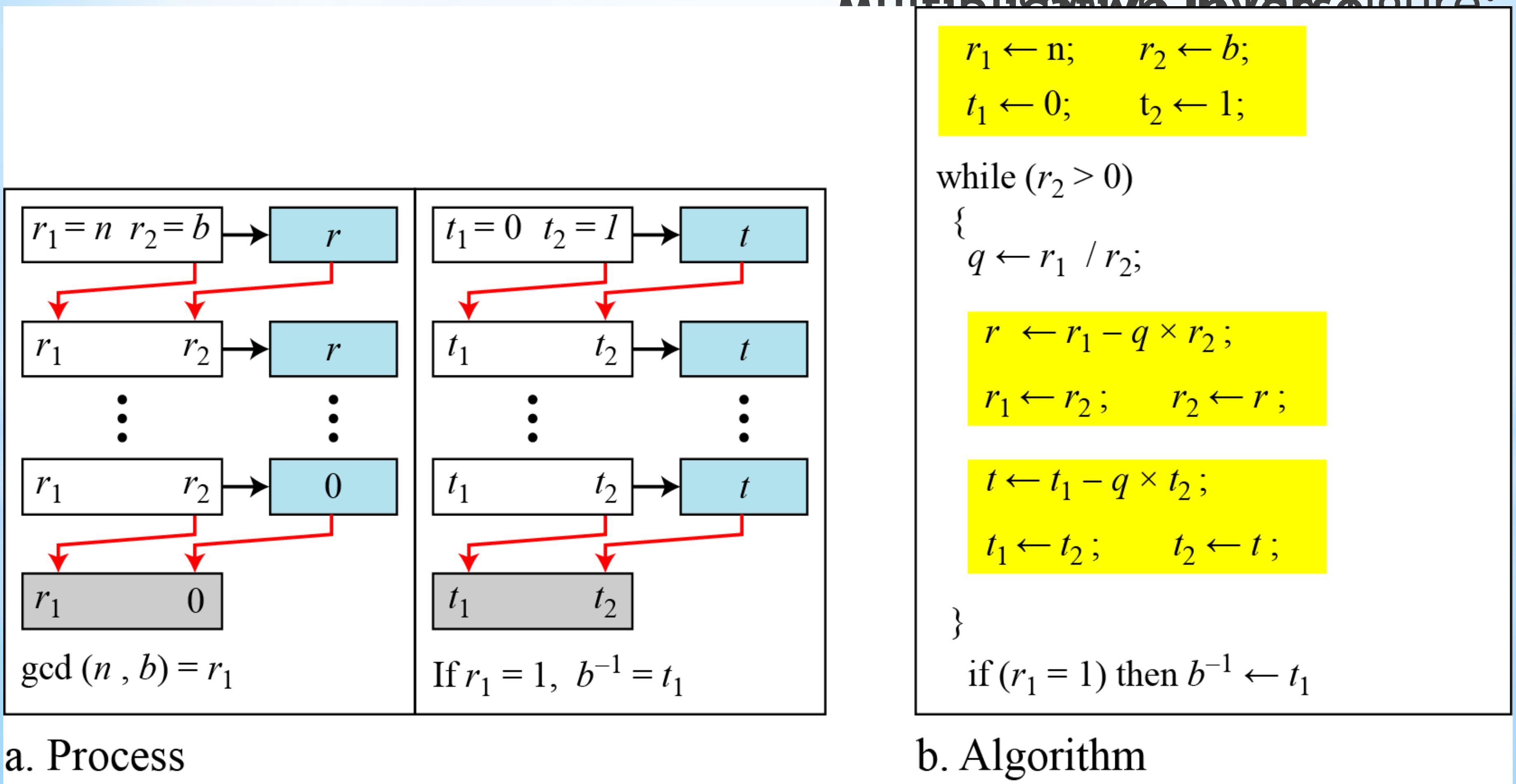
GCD(12,6)=GCD(6, 127 mod 6)=GCD(6,0)=6

LCM(354,143)=(354*144)/6=8496

In Z_n, two numbers a and b are the multiplicative inverse of each other if

$$a \times b \equiv 1 \pmod{n}$$

The extended Euclidean algorithm finds the multiplicative inverses of b in Zn when n and b are given and gcd (n, b) = 1 as shown in this figure:



a. Process

b. Algorithm

Example: - Find the multiplicative inverse of 11 in Z_{26}. *

The GCD(26,11)must be 1 in order to find the inverse. Bu using the extended Euclidean algorithm, we can use this table *

q	r_1	r_2	r	t_1	t_2	t
2	26	11	4	0	1	−2
2	11	4	3	1	−2	5
1	4	3	1	−2	5	−7
3	3	1	0	5	−7	26
	1	0		−7	26	

−7 mod 26=19. *

the equation

$$n = qn + r$$

Example: - Find the multiplicative inverse of 11 in Z_{26}.*

- 26=11*2+4
- 11=4*2+3
- 4=3*1+1
- 3=3*1+0
- We are now in reverse compensation starting from one as shown
- 1=4-(3*1)
- 1=4-(11-(4*2))
- 1=4-11+4*2
- 1=3*4-11
- 1=3*(26-11*2)-11
- 1=3*26-6*11-11= 3*26-7*11 so the multiplicative inverse of 11 is -7

Example :- Find the multiplicative inverse of 23 in Z_{100}.

100=23*4+8

23=8*2+7

8=7*1+1

7=1*7+0

Now in revers way

1=8-(7*1)

1=8-(23-8*2)

1=8-23+8*2

1=3*8-23

1=3*(100-23*4)-23=3*100-12*23-23=3*100**-13***23 So the multiplicative inverse of 23 in Z_{100} is -23 or 87(-23 mod 100).