Chapter three : Mathematical Expectation and Variance

محاضرة (5)
a27 → a34

Def :

if X is discrete r.v. and $f(x)$ is the value of it's probability dist. at x, the expected value of X is :

$$\mu = E(X) = \sum x f(x)$$

And if X is continuous r.v. and $f(x)$ is the value of it's prob. density at x, the expected value of X is :

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

The properties of expectation :
For any constant (a, b)

1. $E(a) = a$ and $E(aX) = aE(X)$
2. $E(aX + b) = aE(X) + b$
3. $E(X + Y) = E(X) + E(Y)$

Ex's /

1. Let X be a r.v. with prob. fun.

X	0	1	2
$f(x)$	$\frac{12}{22}$	$\frac{9}{22}$	$\frac{1}{22}$

Find $E(x)$

Sol /

$$E(x) = \sum x f(x)$$
$$= 0\left(\frac{12}{22}\right) + 1\left(\frac{9}{22}\right) + 2\left(\frac{1}{22}\right) = \frac{11}{22} = \frac{1}{2}$$

(2) For the Prob. density fun. below find $E(X)$.

$$f(x) = 3X^2 \qquad 0 \leq X \leq 1$$

sol/ $E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_0^1 x(3x^2)\, dx \Rightarrow 3\int_0^1 x^3 dx \Rightarrow 3 \cdot \frac{x^4}{4}\Big|_0^1 = \frac{3}{4}$$

Not: if x is discrete r.v. and $f(x)$ is the value of its prob. dist. at x, the expected value of $g(x)$ is given by:

$$E[g(x)] = \sum_x g(x) f(x).$$

and if x is continuous r.v., then the expected value of $g(x)$ is:

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

(3) If the r.v. X has the p.d.f:

$$f(x) = \begin{cases} 0.2 & x = -1 \\ 0.3 & x = 0 \\ 0.5 & x = 1 \end{cases}$$

find $E(x), E(x^2), E(2x), E(2x+1)$

sol/

$$E(x) = \sum x f(x) \Rightarrow (0.2)(-1) + (0)(0.3) + 1(0.5) = 0.3$$

$$E(x^2) = \sum x^2 f(x) \Rightarrow (0.2)\,1 + (0)(0.3) + 1(0.5) = 0.7$$

$$E(2x) = 2E(x) \Rightarrow 2(0.3) \Rightarrow 0.6$$

$$E(2x+1) = 2E(x) + 1 \Rightarrow 2(0.3) + 1 \Rightarrow 1.6$$

Note: For any dist. there is moment about the mean, the expected value for any r.v. (dis. or conti.) is called the first moment which is denoted by $\mu = E(X)$,
where the variance of X is obtain from the second moment $E(X^2)$ is denoted by $V(X) = \sigma^2$, where σ is standard deviation of X and it is given by:

$$\sigma^2 = E(X^2) - (E(X))^2$$

the Properties of variance is:
For any a, b constants.

① $V(a) = 0$ ② $V(aX) = a^2 V(X)$

③ $V(aX \pm b) = a^2 V(X) \mp 0$

Ex.'s /

① For the p.d.f of X:
$f(x) = 2(1-x) \quad 0 < X < 1$
find $E(X)$ and $V(X)$?

sol/ $E(X) = \int_0^1 x f(x)\, dx \Rightarrow 2\int_0^1 x(1-x)\, dx$

$$\Rightarrow 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 \Rightarrow \frac{1}{3}$$

$\cdot V(X) = E(X^2) - (E(X))^2$

$$E(X^2) = \int_0^1 x^2 f(x)\, dx \Rightarrow 2\int_0^1 x^2(1-x)\, dx$$

$$\Rightarrow 2\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 \Rightarrow \frac{1}{6}$$

$$\Rightarrow V(X) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 \Rightarrow \frac{1}{18}$$

② H.W

For the p.d.f of X below. Find the expected value of X and the variance?

$f(x)$	0.1	0.5	0.4
X	0	1	2

③ Let $f(x) = \frac{x}{6} + k$, $0 \leq x \leq 3$, be p.d.f of X find

① The constant k ② $P(1 \leq x \leq 2)$
③ The dist. fun of $F(2)$ ④ $E(x)$ and $V(x)$.

sol/ ① $\int_0^3 f(x)dx = 1 \Rightarrow \int_0^3 (\frac{x}{6} + k) dx = 1$

$$\Rightarrow \frac{x^2}{12} + xk \Big|_0^3 = 1 \Rightarrow \frac{9}{12} + 3k = 1 \rightarrow k = \frac{1}{12}$$

② $P(1 \leq x \leq 2) = \int_1^2 (\frac{x}{6} + \frac{1}{12}) dx \Rightarrow \frac{x^2}{12} + \frac{x}{12} \Big|_1^2$

$$\Rightarrow \left[\frac{4}{12} + \frac{2}{12}\right] - \left(\frac{1}{12} + \frac{1}{12}\right) \Rightarrow \frac{6}{12} - \frac{2}{12} = \frac{4}{12} = \frac{1}{3}$$

③ $F(x) = \int_0^x f(t)\,dt \Rightarrow \int_0^x (\frac{t}{6} + \frac{1}{12}) dt = (\frac{t^2}{12} + \frac{t}{12})\Big|_0^x$

$$\Rightarrow \frac{x^2 + x}{12} \Rightarrow F(2) = \frac{4+2}{12} = \frac{1}{2}$$

④ $E(x) = \int_0^3 x f(x) dx \Rightarrow \int_0^3 x(\frac{x}{6} + \frac{1}{12}) dx$

$$= \frac{1}{6} \int_0^3 (x^2 + \frac{x}{2}) dx = \frac{1}{6}\left[\left(\frac{x^3}{3} + \frac{x^4}{4}\right)\right]_0^3$$

$$= \frac{1}{6}\left(\frac{27}{3} + \frac{9}{4}\right) = 3.75$$

$V(x) = E(x^2) - (E(x))^2 \Rightarrow E(x^2) = \int x^2 (\frac{x}{6} + \frac{1}{12}) dx$

$$\Rightarrow \int (\frac{x^3}{6} + \frac{x^2}{12}) dx \Rightarrow \left[\frac{x^4}{24} + \frac{x^3}{36}\right]\Big|_0^3$$

$\Rightarrow \frac{81}{24} + \frac{27}{36} = 24.75$ $\quad \therefore V(x) = 24.75 - (3.75)^2$
$\Rightarrow 10.69$