Moment generating Function

The moment generating function of a random variable X is given by:

$$M_X(t) = E(e^{tx}) = \sum_x e^{tx} f(x) \quad \text{dis.}$$

and

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \quad \text{con.}$$

Note: we can find the moments from the m.g.f. as:

$$\mu_X = E(X) = \frac{d M_X(t)}{dx}\Big|_{t=0} \quad \left(\text{i.e } E(X) = M'_X(t)\Big|_{t=0}\right)$$

$$E(X^2) = M''_X(t)\Big|_{t=0} \Rightarrow V(X) = E(X^2) - (E(X))^2$$

Ex. ①: Let the r.v. X has the p.d.f:

$$f(x) = \frac{C^3_x}{8} \qquad x = 0, 1, 2, 3$$

Find the moment generating function and use it to find E(X) and V(X)

sol/ $M_X(t) = E(e^{tX}) = \sum e^{tx} f(x) \Rightarrow \frac{1}{8} \sum e^{tx} C^3_x$

$$\Rightarrow \frac{1}{8}\left[e^0 \cdot C^3_0 + e^t \cdot C^3_1 + e^{2t} \cdot C^3_2 + e^{3t} \cdot C^3_3\right]$$

$$\Rightarrow \frac{1}{8}\left[1 + 3e^t + 3e^{2t} + e^{3t}\right] \Leftarrow M_X(t)$$

$$E(X) = M'_X(t)\Big|_{t=0} = \frac{1}{8}\left[0 + 3e^t + 6e^{2t} + 3e^{3t}\right] \Rightarrow \frac{1}{8}(3+6+3)\Big|_{t=0} = \frac{3}{2}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = M_X''(t) = \frac{1}{8}\left(3e^{t} + 12e^{2t} + 9e^{3t}\right)\Big|_{t=0}$$

$$= \frac{1}{8}(3+12+9) \Rightarrow 3$$

$$\Rightarrow V(X) = 3 - \frac{9}{4} \Rightarrow \frac{3}{4}$$

(2) For the m.g.f of the r.v. X below. Find the mean and the variance.

$$M_X(t) = 3(4-e^t)^{-1}$$

sol/ $E(X) = M_X'(t) \Rightarrow -3(4-e^t)^{-2}(-e^t)\Big|_{t=0} \Rightarrow \frac{1}{3}$

$$E(X^2) = M_X''(t) \Rightarrow -3\left[(4-e^t)^{-2}(-e^t) + (-e^t) - 2(4-e^t)^{-3}(-e^t)\right]$$

$$\Rightarrow -3\left[(4-1)^{-2}(-1) + (-1)(-2)(4-1)^{-3}(-1)\right] \Rightarrow \frac{5}{9}$$

$$V(X) = \frac{5}{9} - \frac{1}{9} = \frac{4}{9}$$

(3) H.W

From the m.g.f below find $V(X)$

$$M_X(t) = \frac{\lambda}{\lambda - t} \quad ?$$

(4) Let $E(X-1) = 9$, $E(X^2-5) = 107$. find

$E(X-10)^2$ and $V(15+3C)$ where C any const.

sol/ $E(X-1) = 9 \Rightarrow E(X) = 10$

$E(X^2-5) = 107 \Rightarrow E(X^2) = 112$

$$E(X-10)^2 = E X^2 - 20\, E X + 100$$

$$= 112 - 20(10) + 100 \Rightarrow 12$$

$V(15+3C) = 0$

Ex Find the constant k for the p.d.f and find $P(2 \le X < 5)$, $E(x)$, $V(x)$ and $M_x(t)$

$$f(x) = k(x+2) \qquad x = 0, 1, 2, 3, 4, 5$$

sol/ $\sum f(x) = 1 \Rightarrow 2k + 3k + 4k + 5k + 6k + 7k = 1 \Rightarrow k = \frac{1}{27}$

$$\Rightarrow f(x) = \frac{x+2}{27} \qquad x = 0, 1, 2, 3, 4, 5$$

$$P(2 \le X < 5) = f(2) + f(3) + f(4)$$

$$= \frac{4}{27} + \frac{5}{27} + \frac{6}{27} \Rightarrow \frac{15}{27}$$

$$E(x) = \sum x f(x) \Rightarrow 1 \cdot \frac{3}{27} + 2 \cdot \frac{4}{27} + 3 \cdot \frac{5}{27} + 4 \cdot \frac{6}{27} + 5 \cdot \frac{7}{27} \Rightarrow \frac{85}{27}$$

$$E(x^2) = \sum x^2 f(x) \Rightarrow 1 \cdot \frac{3}{27} + 4 \frac{4}{27} + 9 \cdot \frac{5}{27} + 16 \cdot \frac{6}{27} + 25 \cdot \frac{7}{27} = \frac{335}{27}$$

$$V(x) = Ex^2 - (Ex)^2 \Rightarrow \frac{335}{27} - \frac{7225}{729} = \frac{1739}{729} = 2.497$$

$$M_x(t) = E(e^{tx}) = \sum e^{tx} f(x) \Rightarrow \sum e^{tx} \cdot \frac{x+2}{27}$$

$$= e^0 \cdot \frac{2}{27} + e^t \cdot \frac{3}{27} + e^{2t} \cdot \frac{4}{27} + e^{3t} \cdot \frac{5}{27} + e^{4t} \cdot \frac{6}{27} + e^{5t} \cdot \frac{7}{27}$$

$$\Rightarrow M_x(t) = \frac{1}{27} \left[2 + 3e^t + 4e^{2t} + 5e^{3t} + 6e^{4t} + 7e^{5t} \right)$$

H.W.'S

(1) Let X be a r.v. having the prob. dist. fun.

$f(x) = \frac{1}{2} x \qquad 0 \le x \le 2$

Find $P(1 \le X \le 1.5)$ and the variance of X

(2) For the prob. dist. fun. below, find the expected value of X and the variance from the moment generating function

x_i	1	3	4	5
$f(x)$	0.4	0.1	0.2	0.3

(3) Let $f(x) = k + \frac{1}{6} x \qquad 0 \le x \le 3$ be p.d.f:

(a) - find K

(b) - find $P(1 \le X \le 2)$

(c) - find $E(x)$ and $V(x)$

(4) A box contains 8 units which two of them are defective. three units selected randomly find the expected number of defective units which selected.

(5) Find the moment generating function of the random variable whose p.d.f is given by

$f(x) = e^{-x} \qquad x > 0$

(6) If the random variable X has mean μ and standard deviation σ and $Z = \frac{X-\mu}{\sigma}$

Find $E(Z)$ and $V(Z)$.