Definition:-

Let X be a real or complex vector space over F where F is a field of real number R or complex number $\mathbb{C}$ A mapping $<\ ,\ > : X \times X \rightarrow F$ is called inner product on X . if it's satisfy the following properties :

1- $< x,y> = < y,x> \quad \forall\ x,y \in X.$
2- $< x+y ,z> = <x,z> + < y,z> \quad \forall\ x,y,z \in X$.
3- $< \lambda x ,y> = \lambda<x,y> \quad \forall\ x,y \in X ,\ \lambda\in F$.
4- $<x,x> \ > 0$ when $X\neq 0$.

Definition:-

An inner product space is a vector space X with inner product defined on X . Then (x, < , >) is inner product space.

Example:-

Let $X= \mathbb{C}^n$, The set of all n- tuples of complex number $X= (\alpha_1,\alpha_2 ,\ldots, \alpha_n)$,

$y =(\beta_1,\beta_2,\ldots,\beta_n)$ where α_ι, β_ι are complex number define

$<x,y> = \sum_{i=1}^{n} \alpha_\iota \overline{\beta_\iota}$ Then the order pair ($\mathbb{C}^n$, < , >) is an inner product space.

Solution:-

1- $<x,y> = \sum_{i=1}^{n} \propto \iota \overline{\beta} \iota$

$\overline{< y,x >} = \overline{\sum_{\iota=1}^{n} \beta\iota\ \overline{\alpha\iota}} = \sum_{i=1}^{n} \overline{\beta\iota}\ \overline{\overline{\alpha}\iota} = \sum_{i=1}^{n} \overline{\beta\iota}\ \alpha\iota = \sum_{i=1}^{n} \propto \iota\overline{\beta}\iota$

$\therefore < x,y> =\overline{< y,x >}$.

2- $< x+y , z> = \sum_{i=1}^{n}(\alpha\iota + \beta\iota)\overline{\gamma\iota}$ Where $Z \in \mathbb{C}^n$, $z = (\gamma_1, \gamma_2, \ldots, \gamma_n)$

$= \sum_{i=1}^{n}(\alpha\iota\, \overline{\gamma\iota} + \beta\iota\, \overline{\gamma\iota}) = \sum_{i=1}^{n} \alpha\iota\ \overline{\gamma\iota} + \sum_{i=1}^{n} \beta\iota\ \overline{\gamma\iota}$

$= < x,z> + <y,z>$.

3- $< \lambda x,y> = \sum_{i=1}^{n} \lambda\, \alpha\iota\, \overline{\beta\iota}$

$= \lambda \sum_{i=1}^{n} \alpha\iota\, \overline{\beta\iota}$

$= \lambda < x,y>$

4- $< x,x> = \sum_{i=1}^{n} \propto \iota\ \overline{\alpha\iota}$

$= \sum_{i=1}^{n} \left| \propto \iota \right|^2 > 0 \longrightarrow < x , x> > 0$

When $x \neq \theta$

$\therefore$ ($\mathbb{C}$, < , >) is inner product space.

Example:-

Let X= c[a,b] , The set of all continuous function defined on closed interval [a,b] with vector addition $(f+g)_{(x)} = f(x) + g(x)$.

Scalar multiplication $(\alpha f)_{(x)} = \alpha f(x)$.

Define $<f,g> = \int_a^b f(x) . \overline{g(x)}\, dx$ then (x, < ,>) is an inner product space .

Solution:-

1- $<f ,g> = \int_a^b f(x) . \overline{g(x)}\, dx$

$\overline{<g,f>} = \overline{\int_a^b g(x)\ . \overline{f(x)}\ dx}$

$= \int_a^b \overline{g(x)}\ . \overline{\overline{f(x)}}\ dx$

$= \int_a^b \overline{g(x)}. f(x)\ dx$

$= \int_a^b f(x) . \overline{g(x)}$ dx

$\therefore$ <f ,g> = $\overline{< g, f >}$.

2- $< f+g , h > = \int_a^b ((f + g)_{(x)}) . \overline{h(x)}$ dx

$= \int_a^b (f(x) + g(x)) . \overline{h(x)}$ dx

$= \int_a^b f(x) . \overline{h(x)}$ dx $+ \int_a^b g(x) . \overline{h(x)}$ dx

$= <f, h> + <g, h>$.

3- $<\lambda f , g> = \int_a^b (\lambda f)_{(x)} . \overline{g(x)}$ dx $= \lambda \int_a^b f(x) . \overline{g(x)}$ dx

$= \lambda < f, g >$.

4- $< f , f >$ when $f \neq 0$

$< f, f> = \int_a^b f(x) . \overline{f(x)}\ dx$

$= \int_a^b |f(x)|^2$ dx > 0

$\therefore <f ,f> >0$

$\therefore$ (x , < , >) is inner product space.

Remark:-

Let X be inner product space then any subspace of X is also inner product space.

Theorem:-

In any inner product space then :

1- $< x,y+z > = <x,y> + <x,z>$.

2- $< x, \lambda y> = \bar{\lambda} < x,y>$.

3- $<\theta ,y> = < x,\theta> =0$.

4- $<x-y,z> = < x,z> - < y,z >$.

5- $< x ,y-z > = < x,y> - < x, z >$.

6- $< x,z> = < y, z>$ for all z then x=y.

Proof :-

1- $< x,y+z > \overline{< y + z , x >} = \overline{< y , x > + < z, x >}$

$= \overline{<y, x >} + \overline{<z + x >} = <x,y>+<x,z>$.

2- $<x,\lambda y> = \overline{<\lambda y, x >} = \bar{\lambda}\, \overline{<y, x >} = \bar{\lambda} <x, y >$.

3- $<\theta, y> = <\theta +\theta,y> = <\theta,y>+<\theta,y>$

$\Rightarrow <\theta,y> - <\theta,y>= <\theta,y>$

$\Rightarrow 0 = <\theta,y>$

$\Rightarrow <\theta,y> =0$

$< x,\theta> = < x,\theta+\theta> = <x,\theta> + <x,\theta>$

$\Rightarrow <x,\theta> - <x,\theta> = <x,\theta>$

$\Rightarrow 0= <x,\theta>$

$\Rightarrow <x,\theta> =0$.

4- $< x-y ,z> = < x+(-y) ,z> = <x,z> + <-y, z>$

$= <x,z> - <y ,z>$.

5- $< x,y-z> = <x ,y+(-z)> = <x,y> + <x,-z>$

$= <x,y> - <x,z>.$

6- $<x,z> = <y,z>$

$\Rightarrow <x,z> - <y,z> = 0$

$\Rightarrow < x-y,z> = 0$ for all z

$\Rightarrow x-y = 0$

$\Rightarrow x=y$.