1.4 Banach space

Definition(Banach space):- Let X be a normed linear space , Let d be a metric space defined on X if (X,d) is complete then X is called Banach space.

Remark:- Complete normed vector space = Banach space.

Example:- Let X be a vector space IR or $\mathbb{C}$ for $x \in X$ Tack $\|x\| = |x|$ Then $(X, \|\ \|)$ is Banach space.

Sol:-

1) $\|x\| = |x| \geq 0$
2) $\|x\| = 0 \rightarrow |x| = 0 \rightarrow x = 0$

$X = 0 \rightarrow |x| = 0 \rightarrow \|x\| = 0$

3)Let $z, w \in \mathbb{C}$

$\|z+w\| = |z+w|$

$|z+w|^2 = (z+w)(\overline{z+w})$

$= (z+w)\cdot(\bar{z}+\bar{w})$

$= z\cdot\bar{z} + z\bar{w} + \bar{z}w + w\cdot\bar{w}$

$\leq |z|^2 + 2|z\bar{w}| + |w|^2$ by $|z|^2 = z\cdot\bar{z}$

$\leq |z|^2 + 2|z|\cdot|w| + |w|^2$ by $|z\bar{w}| = |zw| = |z|\cdot|w|$

$\leq (|z|+|w|)^2$

hence $|z+w| \leq |z|+|w| \rightarrow \|z+w\| \leq \|z\|+\|w\|$.

4) $\|\lambda x\| = |\lambda x| = |\lambda|\cdot|x| = |\lambda|\ \|x\|$

Thus $\mathbb{C}$ and R are both normed linear space and since every Cauchy sequence in $\mathbb{C}$ or (IR) is convergent , Then $\mathbb{C}$ and IR are complete.

hence $\mathbb{C}$ and IR are Banach space.

Example:- (Euclidean and unitary spaces)

Show that the linear space $\mathbb{R}^n$ and $\mathbb{C}^n$ of all n-tuples

$X = (x_1,x_2,\dots,x_n)$, $x_1,x_2,\dots,x_n \in \mathbb{R}^n$ or $\mathbb{C}$ are Banach space under the norm

$\|x\|=\left(\sum_{i=1}^{n} |xi|^2\right)^{\frac{1}{2}}$

Sol:-

1. Since $|x_i| \geq 0 \quad \forall\ i=1,\dots,n \rightarrow \sum_{i=1}^{n} |xi|^2 \geq 0$

$\rightarrow \left(\sum_{i=1}^{n} |xi|^2\right)^{\frac{1}{2}} \geq 0$

$\rightarrow \|x\| \geq 0.$

2. if $\|x\| = 0 \rightarrow \left(\sum_{i=1}^{n} |xi|^2\right)^{\frac{1}{2}} = 0 \rightarrow \sum_{i=1}^{n} |xi|^2 = 0 \rightarrow x_i = 0 \quad \forall i=1,\dots,n$

if $x_i = 0 \quad \forall i=1,\dots,n$

$\rightarrow |x_i|^2 = 0 \rightarrow \sum_{i=1}^{n} |xi|^2 = 0$

$\rightarrow \left(\sum_{i=1}^{n} |xi|^2\right)^{\frac{1}{2}} = 0$

$\rightarrow \|x\| = 0.$

3. $\|x+y\|^2 = \|(x_1+y_1,x_2+y_2,\dots,x_n+y_n)\|^2$

$= \sum_{i=1}^{n} |xi+yi|^2$

$= \sum_{i=1}^{n} |xi+yi|\,|xi+yi|$

$\leq \sum_{i=1}^{n} |xi+yi|\,(|xi|+|yi|)$

$\leq \sum_{i=1}^{n} (|xi+yi|)\,|xi| + \sum_{i=1}^{n} (|xi+yi|)\,|yi|$

$= \|x+y\|\cdot\|x\| + \|x+y\|\cdot\|y\|$

$= \|x+y\|\,(\|x\|+\|y\|)$

if $\|x+y\| = 0$, then the above is true

if $\|x+y\| \neq 0 \rightarrow \|x+y\| \leq \|x\|+\|y\|$.

4. $\|\lambda x\| = \|(\lambda x_1,\lambda x_2,\dots\dots,\lambda x_n)\| = \left(\sum_{i=1}^{n} |\lambda xi|^2\right)^{\frac{1}{2}}$

$= \left(\sum_{i=1}^{n} |\lambda|^2 |xi|^2\right)^{\frac{1}{2}}$

$= \left(|\lambda|^2 \sum_{i=1}^{n} |xi|^2\right)^{\frac{1}{2}}$

$= |\lambda| \left(\sum_{i=1}^{n} |xi|^2\right)^{\frac{1}{2}}$

$=|\lambda|\ ||x||$.

Now to prove $\mathbb{R}^n$ is complete

Let $<X_n> = <x_1,x_2,\ldots,x_n>$ be a Cauchy sequence in R^n where

$<X_1> = <a_1^{(1)}, a_1^{(2)},\ldots, a_1^{(n)}>$

$<X_2> = <a_2^{(1)}, a_2^{(2)},\ldots, a_2^{(n)}>$

The projection X_n in $\mathbb{R}$

$<a_1^{(1)}, a_2^{(1)},\ldots>, <a_1^{(2)}, a_2^{(2)},\ldots>,\ldots, <a_1^{(n)}, a_2^{(n)},\ldots>$ is Cauchy in $\mathbb{R}$ and since $\mathbb{R}$ is complete.

Then $<a_1^{(1)}, a_2^{(1)},\ldots>, <a_1^{(2)}, a_2^{(2)},\ldots>,\ldots, <a_1^{(n)}, a_2^{(n)},\ldots>$ are convergent.

$<a_1^{(1)}, a_2^{(1)},\ldots>\to b_1,\ldots, <a_1^{(n)}, a_2^{(n)},\ldots>\to b_n\in\mathbb{R}^n$

Then $X_n \to q = < b_1,b_2,\ldots,b_n>$

$\therefore X_n$ convergent $\to \mathbb{R}^n$ is complete.

$\therefore \mathbb{R}^n$ Banach space.

Example:-

Let C(x) denote the linear space of all bounded continuous mapping on space X, Show that C(x) is a Banach space under the norm

$||f||= \text{Sup}\ \{|f(x)|:x\in X\}\ ,f\in C(x)$.

Sol:-

Let $(f+g)(x)= f(x)+g(x)$

$(\alpha f)(x) = \alpha f(x)$.

1.since $|f(x)| \geq 0 \quad \forall x\in X \ \to ||f|| \geq 0$.

2.$||\ f|| = 0 \Leftrightarrow \text{Sup}\ \{|f(x)|:x\in X\} = 0$

$\Leftrightarrow |f(x)| = 0 \quad \forall x\in X$

$\Leftrightarrow f(x) = 0 \quad \forall x\in X \ \Leftrightarrow f= \hat{0}$ (zero function).

3. $\| f+g\| = \text{Sup}\{|(f+g)(x)|:x\in X\}$

$= \text{Sup}\{|f(x) + g(x)|:x\in X\}$

$\leq \text{Sup}\{|f(x)|:x\in X\}+\text{Sup}\{|g(x)|:x\in X\}$

$\leq \|f\| + \| g\|$.

4. $\|\alpha f\| = \text{Sup}\{|(\alpha f)(x)|:x\in X\}$

$= \text{Sup}\{|\alpha f(x)|:x\in X\}$

$= \text{Sup}\{|\alpha|\cdot|f(x)|:x\in X\}$

$= |\alpha| \text{Sup}\{|f(x)|:x\in X\}$

$= |\alpha| \|f\|$.

Hence C(x) is normed linear space

Now to prove C(x) is complete.

Let $\{f_n\}^{\infty}_{n=1}$ be any Cauchy sequence in C(x) then ,$\forall\varepsilon > 0$ $\exists m_o > 0$

Such that $\forall m ,n \geq m_o \rightarrow \|f_m-f_n\| < \varepsilon$

$\rightarrow \text{Sup}\{|(f_m - f_n)(x)|:x\in X\} < \varepsilon$

$\text{Sup}\{|f_m(x) -f_n(x)|: x\in X\} < \varepsilon$

$\rightarrow|f_m(x)-f_n(x)| < \varepsilon \quad \forall x \in X$

This condition of uniform convergent.

Hence $\{f_n\}^{\infty}_{n=1}$ must convergent to bounded continuous function f on x

i.e $f_n \rightarrow f\in C(x)$.

Thus C(x) is complete and hence if is Banach space.

Exe:-

1) Let $X \neq \emptyset$ be any set , and let $\beta(X)=\{f/\ f: x \to R \text{ is bounded}\}$

Show that $\beta(x)$ becomes a normed linear space if we define

$\|f\| = \text{Sup}\{|f(x)|: x \in X\},\ f \in \beta(x)$.

2) Let $C[0,1]=\{f/f:[0,1] \to \mathbb{R} \text{ is continuous}\}$
Defined a normed by $\|f\| = \max\{|f(t)|\}$, $0 \leq t \leq 1$.

Remark:-
Every Banach space is a normed space but the converse is not necessary to be true we can see the following example.

Example:-
Let X be a normed space of finitely non-zero sequence with $d(x,y) = \|x-y\|$, Show that X is in complete .(not Banach space).
Sol:-
$X_1= (1,0,0,\ldots)$
$X_2= (1,\frac{1}{2},0,0,\ldots)$
$X_3= (1,\frac{1}{2},\frac{1}{3},0,0,\ldots.)$
:
:
$X_n= (1,\frac{1}{2},\frac{1}{3},\ldots\ldots,\frac{1}{n},0,0,\ldots.)$
For all n ,p =1,2,3
$\|x_{n+p}- x_n\|^2 = \|(0,0,\ldots,\frac{1}{n+1},\frac{1}{n+2},\ldots..,\frac{1}{n+p},0,0,\ldots.)\|^2$
$= \sum_{n+1}^{n+p} \frac{1}{k^2}$
Since the sequence is convergent
$\to d(x_{n+p}, x_n) = \|x_{n+p}- x_n\| \to 0$
As $n\to\infty$, that is x_n is Cauchy sequence
Suppose X contained a vector $X= (\lambda_1,\lambda_2,..,\lambda_n,0,0,\ldots)$
Such that $x_n \to x$

If $n \geq N$

$\|x_n - x\|^2 = \sum_1^n |\frac{1}{k} - \lambda_k|^2$

Letting $n \to \infty \to \sum_1^n |\frac{1}{k} - \lambda_k|^2 = 0$,

hence $\lambda_k = \frac{1}{k}$ for all k

This contradiction that x is non –finitely zero.