Discrete Transform

6-1 Introduction

A transform is simply another term for a mathematical mapping process. there are some of the transforms are used in image analysis and processing to provide information regarding the rate at which the gray levels change within an image—the spatial frequency. However, its primary purpose is to decorrelate the data between image bands. Additionally, the wavelet and the Haar transforms are different in that they retain both spatial and frequency information.

In general, a transform maps image data into a different mathematical space via a transformation equation. However, those color transforms mapped data from one color space to another color space with a one-to-one correspondence between a pixel in the input and the output. Basically, most of these transforms map the image data from the spatial domain to the frequency domain (also called the spectral domain), where all the pixels in the input (spatial domain) contribute to each value in the output (frequency domain). (Figure 6.1)

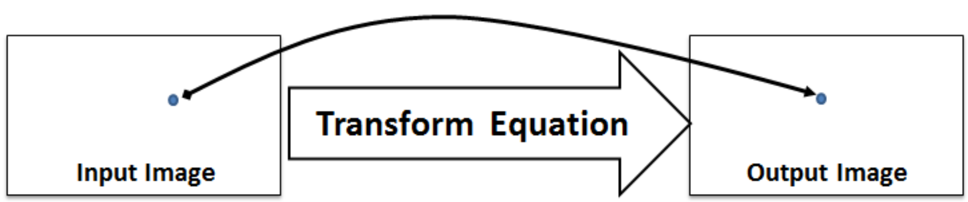

Figure 6.1: Color transforms use a single-pixel to single-pixel mapping.

These transforms are used as tools in many areas of engineering and science, including digital image processing. Originally defined in their continuous forms, they are commonly used today in their discrete (sampled) forms. The large number of arithmetic operations required for the discrete transforms, combined with the massive amounts of data in an image, require a great deal of computer power. The ever-increasing computer power, memory capacity, and disk storage available today make the use of these transforms much more feasible than in the past.(figure 6.2)

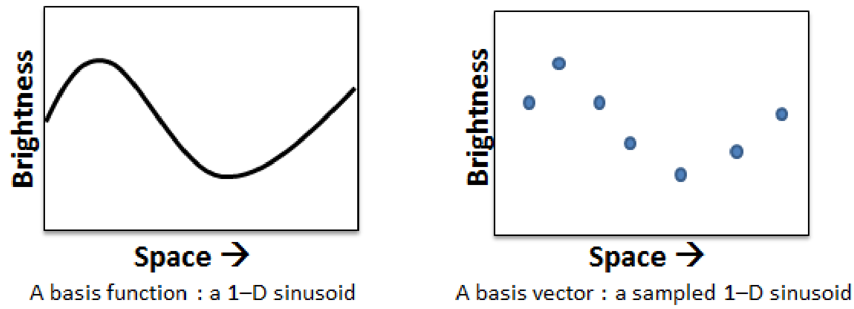

Figure 6.2: The pixel brightness in each row corresponds to 1–D sinusoids, which are repeated along each column for 2-D Image.

Image transforms are designed to reduce image redundancy by reducing the sizes of most pixels and to identify the less important parts of the image by isolating the various frequencies of the image.

Image frequencies are important because the Low frequencies correspond to the important image features, whereas high frequencies correspond to the details of the image, which are less important. Thus, when a transform isolates the various image frequencies, pixels that correspond to high frequencies can be quantized heavily, whereas pixels that correspond to low frequencies should be quantized lightly or not at all. This is how a transform can compress an image very effectively by losing information, but only information associated with unimportant image details.

6-2Fourier Transform (FT)

The Fourier transform is the most well-known, and the most widely used transform. It was developed by Baptiste Joseph Fourier (1968-1830) to explain the distribution of temperature and heat conduction. Since that time the Fourier transform has found numerous uses, including vibration analysis in mechanical engineering, and here in computer imaging. In mathematics, a Fourier series decomposes periodic functions or periodic signals into the sum of a (possibly infinite) set of simple oscillating functions, namely sine and cosine (or complex exponentials).

This transform allows for decomposition of an image into weighted sum of 2-D sinusoidal terms. Assuming an N×N image, the equation for the 2-D discrete Fourier Transform (2-D DFT).

$$F(u,v) = \frac{1}{N}\sum_{r=0}^{N-1}\sum_{c=0}^{N-1} I(r,c)e^{-j2\pi(ur+vc)/N}$$

The base of the natural logarithmic function e is about 2.71828; j, the imaginary coordinates for a complex number, equal $\sqrt{-1}$. The basis functions are sinusoidal in nature, as can be seen by Euler's identity:

$e^{jx} = \cos x + j \sin x$

So we can write the Fourier transform equation as

$$F(u,v) = \frac{1}{N}\sum_{r=0}^{N-1}\sum_{c=0}^{N-1} I(r,c)\left[\cos\left(\frac{2\pi}{N}(ur+vc)\right) - j\sin\left(\frac{2\pi}{N}(ur+vc)\right)\right]$$

In this case, F(u, v) is also complex, with the real part corresponding to the cosine terms and the imaginary part corresponding to the sine terms.

After we perform the transform, if we want to get our original image back, we need to apply the inverse transform. The inverse Fourier transform is given by:

$$F^{-1}[F(u,v)] = I(r,c) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ur+vc)/N}$$

The F^{-1}[ ] notation represents the inverse transform. This equation illustrates that the function, I(r,c), is represented by a weighted sum of the basis functions, and that the transform coefficients, F(u,v), are the weights. With the inverse Fourier transform, the sign on the basis functions' exponent is changed from –1 to +1. (Figure 6.3)

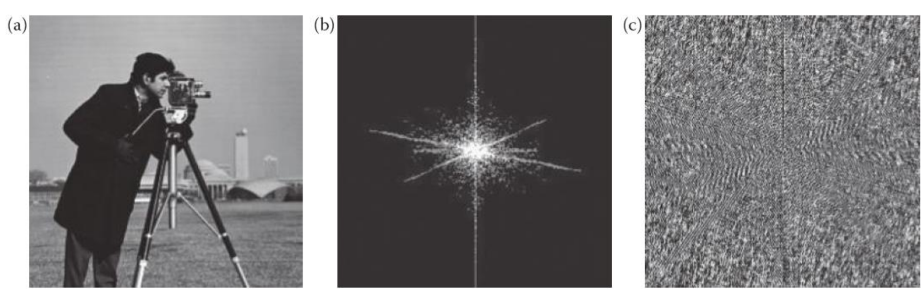

Figure 6.3: a-original image b-the magnitude of the Fourier spectrum from (a) represented as an image, c- the phase of the Fourier spectrum from (a) represented by an image.

6-3Discrete Cosine Transform

The cosine transform, like the Fourier transform, uses sinusoidal basis functions. The difference is that the cosine transform basis functions are not complex; they use only cosine functions, and not sine functions. The 2-D discrete cosine transform (DCT) equation for an N × N image is given by

$$C(u,v) = \alpha(u)\alpha(v)\sum_{r=0}^{N-1}\sum_{c=0}^{N-1} I(r,c)\cos\left[\frac{(2r+1)u\pi}{2N}\right]\cos\left[\frac{(2c+1)v\pi}{2N}\right]$$

Where

$$\alpha(u),\alpha(v) = \begin{cases} \sqrt{\frac{1}{N}}\ for\ u,v = 0 \\ \sqrt{\frac{2}{N}}\ for\ u,v = 1,2,\ldots,N-1 \end{cases}$$

Because this transform uses only the cosine function, it can be calculated using on real arithmetic (not complex).

The cosine transform is often used in image compression, in particular in the first version of the Joint Photographers Expert Group (JPEG) image compression method, which has been established as an international standard (the newer JPEG2000 method uses the wavelet transform). In digital image processing we often represent the basis matrices as images, called basis images, where we use various gray values to represent the different values in the basis matrix.

The important feature of the DCT, the feature that makes it so useful in data compression, is that it takes correlated input data and concentrates its energy in just the first few transform coefficients.

6-4Wavelet Transform

A wavelet is a waveform of effectively limited duration that has an average value of zero. However, wavelet is a “small wave” which has its energy concentrated in time to give a tool for the analysis.

Compare wavelets with sine waves, which are the basis of Fourier analysis. Sinusoids do not have limited duration — they extend from minus to plus infinity. And where sinusoids are smooth and predictable, wavelets tend to be irregular and asymmetric. (Figure 6.4)

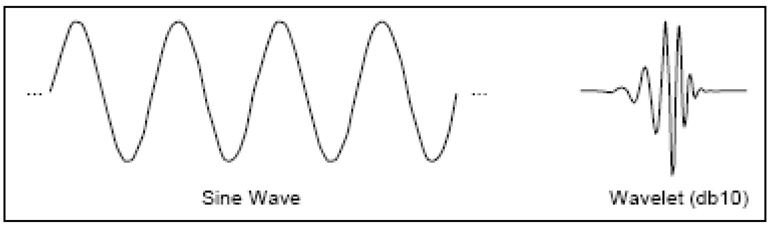

Figure 6.4: sin wave and wavelet represent.

Wavelet analysis represents a windowing technique with variable-sized regions. Wavelet analysis allows the use of long time intervals where we want more precise low-frequency information, and shorter regions where we want high frequency information.

Signals with sharp changes might be better analyzed with an irregular wavelet than with a smooth sinusoid.

- **Scaling**: Scaling a wavelet simply means stretching (or compressing) it. To go beyond colloquial descriptions such as “stretching,” we introduce the scale factor, often denoted by the letter a.

The scale factor works wavelets. The smaller the scale factor, the more “compressed” the wavelet. The effect of the scale factor is very easy to see.(Figure 6.5)

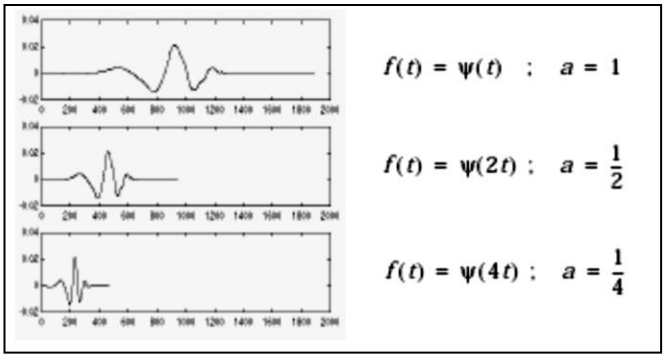

Figure 6.5: Scale Factor with Wavelet.

➢ **Scale and Frequency:** there is a correspondence between wavelet scales and frequency as revealed by wavelet analysis: (Figure 6.6)

• Low scale a=> Compressed wavelet => Rapidly changing details => High frequency 'w'.

• High scale a=>Stretched wavelet=>Slowly changing, coarse features=>Low frequency 'w'.

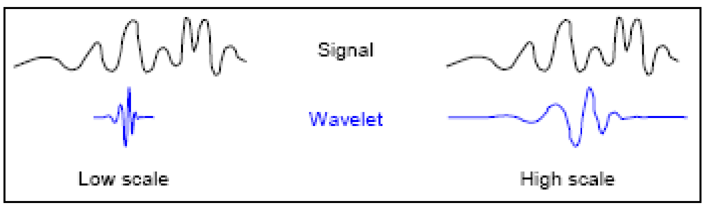

Figure 6.6: Scale and Frequency.

6-4-1 The Discrete Wavelet Transform

It turns out rather remarkably that if we choose scales and positions based on powers of two—so-called dyadic scales and positions—then our analysis will be much more efficient and just as accurate. We obtain such an analysis from the discrete wavelet transform (DWT).

For many signals, the low-frequency content is the most important part. It is what gives the signal its identity. The high-frequency content on the other hand imparts flavor or nuance. Consider the human voice. If you remove the high-frequency components, the voice sounds different but you can still tell what's being said. However, if you remove enough of the low-frequency components, you hear gibberish. In wavelet analysis, we often speak of approximations and details. The approximations are the high-scale, low-frequency components of the signal. The details are the low-scale, high-frequency components. (Figure 6.7)

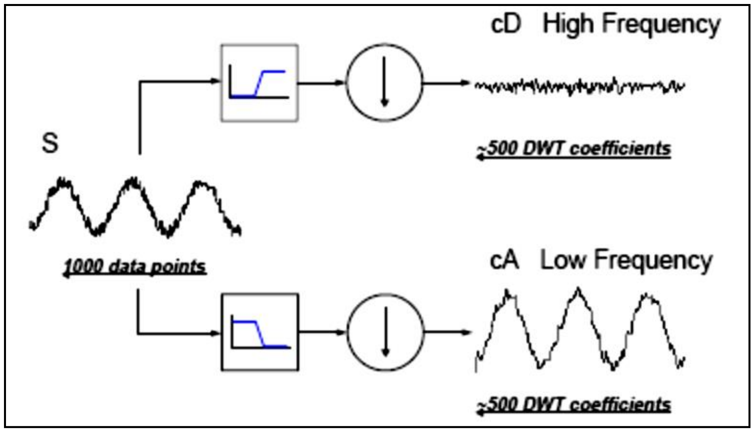

Figure 6.7: Low and High Frequency in DWT.

- **Multiple-Level Decomposition:**

The important of the wavelet as a multiresolution analysis technique comes from its decomposition of the image into multilevel of the independent information with changing the scale like a geographical map in which the image has non – redundant information due to the changing of the scale.

The decomposition process can be iterated, with successive approximations being decomposed in turn, so that one signal is broken down into many lower resolution components. This is called the wavelet decomposition tree.

Using this fact, every image will be transformed in each level of decomposition to a one low information sub-image and three details sub-images in Horizontal, Vertical, and Diagonal axis sub-images, also the low information sub-image can be decomposed into another four sub-images. This approach of decomposition process provides a number of unrealizable features for the original image, which appear in these levels after applying the transformation. Since wavelet transform provides a powerful time – frequency representation then it can be considered as the most efficient transform, which deals with images, sound, or any other pattern. (figure 6.8)

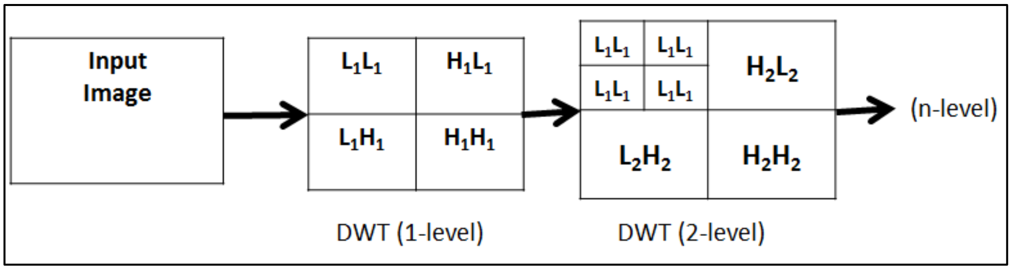

Figure 6.8: Block diagram of n-level wavelet transform

➢ **Number of Levels:** Since the analysis process is iterative, in theory it can be continued indefinitely. In reality, the decomposition can proceed only until the individual details consist of a single sample or pixel. In practice, you'll select a suitable number of levels based on the nature of the signal, or on a suitable criterion such as entropy. (Figure 6.9)

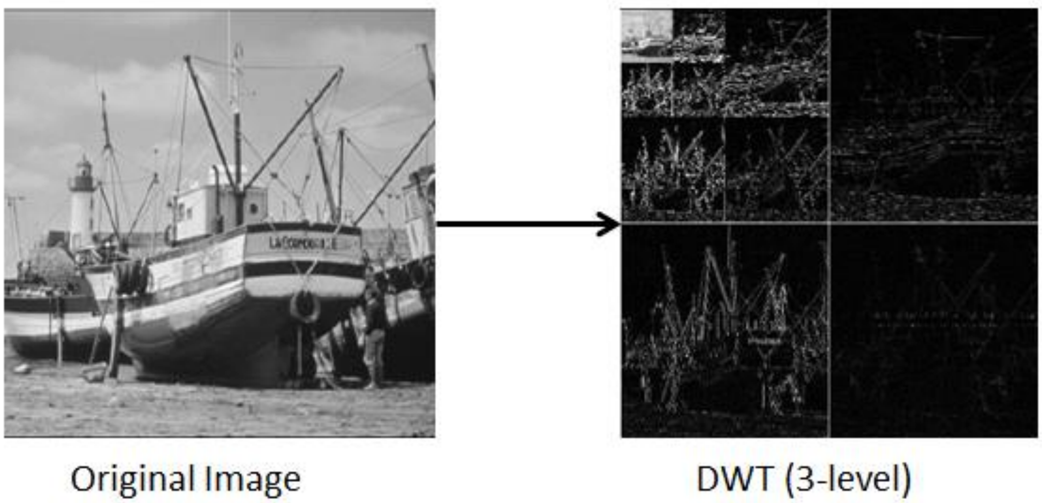

Figure 6.9: Example of Wavelet Transform.

6-4-2 Wavelet Reconstruction:

We've learned how the discrete wavelet transform can be used to analyze or decompose, signals and images. This process is called decomposition or analysis. The other half of the story is how those components can be assembled back into the original signal without loss of information. This process is called reconstruction, or synthesis. The mathematical manipulation that effects synthesis is called the inverse discrete wavelet transforms (IDWT).

Where wavelet analysis involves filtering and down sampling, the wavelet reconstruction process consists of up sampling and filtering. Up sampling is the process of lengthening a signal component by inserting zeros between samples. (Figure 6.10)

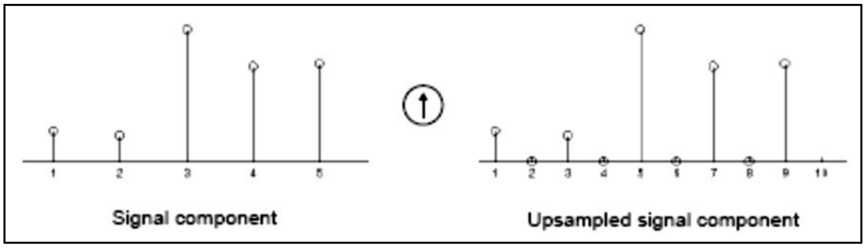

Figure 6.10: Up Sampling Process.

The low- and high pass decomposition filters (L and H), together with their associated reconstruction filters (L' and H'), form a system of what is called <u>quadrature mirror filters</u>.(Figure 6.11)

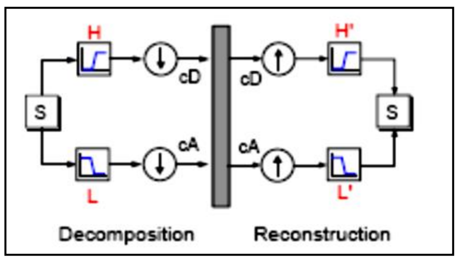

Figure 6.11: quadrature mirror filters.