The Adjoint of a Linear Operator

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Adjoint Operator

Let $L : V \to V$ be a linear operator on an inner product space V.

Definition

The adjoint of L is a transformation $L^* : V \to V$ satisfying

$$\langle L(\vec{x}), \vec{y} \rangle = \langle \vec{x}, L^*(\vec{y}) \rangle$$

for all $\vec{x}, \vec{y} \in V$.

Observation

The adjoint of L may not exist.

Representation of Linear Functionals

Theorem

Let V be a finite-dimensional inner product space over a field F, and let $g : V \to F$ be a linear transformation. Then there exists a unique vector $\vec{y} \in V$ such that $g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x} \in V$.

Proof Idea

Let $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be an orthonormal basis for V and take

$$\vec{y} = \sum_{i=1}^{n} \overline{g(\vec{v}_i)} \vec{v}_i.$$

Warning

In the prior theorem, the assumption that V is finite-dimensional is essential.

Let V be the vector space of polynomials over the field of complex numbers with inner product $\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\, dt$.

Fix a complex number z and let L be the linear functional defined by evaluation at z. That is, take $L(f) = f(z)$ for each f in V. Note that L is not the zero functional.

Claim:
There is no polynomial g such that

$$L(f) = \langle f, g \rangle$$

for all f in V.

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Suppose that we have

$$f(z) = \int_0^1 f(t)\overline{g(t)}\, dt$$

for all f.

Let $h = x - z$, so that for any f we have $(hf)(z) = 0$. Then

$$0 = \int_0^1 h(t)f(t)\overline{g(t)}\, dt$$

for all f.

Warning

Claim:
There is no polynomial g such that

$$L(f) = \langle f, g \rangle$$

for all f in V.

In particular, when $f = \overline{h}g$ we have

$$0 = \int_0^1 |h(t)|^2 \, |g(t)|^2 \, dt$$

so that $hg = 0$.

Since $h \neq 0$, we must have $g = 0$. But then $L(f) = \langle f, g \rangle = 0$ for all f, and we know that L is not the zero functional.

Existence and Uniqueness of the Adjoint Operator

Theorem

Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Then there exists a unique function $T^* : V \to V$ such that $\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T^*(\vec{y}) \rangle$ for all $\vec{x}, \vec{y} \in V$. Furthermore, T^* is linear.

Proof Idea

Let $\vec{y} \in V$. Define $g_{\vec{y}} : V \to F$ by $g_{\vec{y}}(\vec{x}) = \langle T(\vec{x}), \vec{y} \rangle$ for all $\vec{x} \in V$.

Apply the result of the prior theorem to obtain a unique vector $\vec{y}' \in V$ such that $g_{\vec{y}}(\vec{x}) = \langle \vec{x}, \vec{y}' \rangle$ for all $\vec{x} \in V$.

Define $T^* : V \to V$ by $T^*(\vec{y}) = \vec{y}'$.

Observation

Let T be a linear operator on an inner product space V having adjoint T^*.

We have that

$$\langle \vec{x}, T(\vec{y}) \rangle = \overline{\langle T(\vec{y}), \vec{x} \rangle} = \overline{\langle \vec{y}, T^*(\vec{x}) \rangle} = \langle T^*(\vec{x}), \vec{y} \rangle.$$

Properties of Adjoint Operators

Theorem

Let V be a finite dimensional inner product space over a field F, and let T and U be linear operators on V having adjoints. Then

(a) $(T+U)^* = T^* + U^*$

(b) $(cT)^* = \overline{c}T^*$ for any $c \in F$

(c) $(TU)^* = U^*T^*$

(d) $(T^*)^* = T$

(e) $\mathrm{I}^* = \mathrm{I}$

Adjoint Matrix

Let $A = (a_{ij})$ be an $m \times n$ matrix with complex entries.

Definition

The adjoint matrix of A is the $n \times m$ matrix $A^* = (b_{ij})$ such that $b_{ij} = \overline{a_{ji}}$.

That is, $A^* = \overline{A^t}$.

Example

Given $A = \begin{bmatrix} 1 & -2i \\ 3 & i \end{bmatrix}$, note that $A^* = \begin{bmatrix} 1 & 3 \\ 2i & -i \end{bmatrix}$.

Properties of Adjoint Matrices

Corollary

Let A and B be $n \times n$ matrices. Then

(a) $(A + B)^* = A^* + B^*$

(b) $(cA)^* = \overline{c}A^*$ for all $c \in F$

(c) $(AB)^* = B^*A^*$

(d) $(A^*)^* = A$

(e) $\mathrm{I}^* = \mathrm{I}$

The Matrix of the Adjoint Operator

Theorem

Let V be a finite-dimensional inner product space, and let β be an orthonormal basis for V. If T is a linear operator on V, then $[T^*]_\beta = [T]_\beta^*$.

Proof

Let $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$, let $A = (a_{ij})$ be the matrix of T, and let $B = (b_{ij})$ be the matrix of T^*.

Since β is orthonormal, we have $a_{ij} = \langle T(\vec{v}_j), \vec{v}_i \rangle$ and $b_{ij} = \langle T^*(\vec{v}_j), \vec{v}_i \rangle$.

Observe that

$$b_{ij} = \langle T^*(\vec{v}_j), \vec{v}_i \rangle = \overline{\langle \vec{v}_i, T^*(\vec{v}_j) \rangle} = \overline{\langle T(\vec{v}_i), \vec{v}_j \rangle} = \overline{a_{ji}},$$

from which $B = A^*$ follows.

Example

Let $V = \mathbb{C}^2$ with inner product

$$\langle \vec{x}, \vec{y} \rangle = x_1 \overline{y_1} + x_2 \overline{y_2}.$$

Define a linear operator L on V by

$$L(z_1, z_2) = (z_1 - 2i\, z_2, 3z_1 + i\, z_2).$$

Find the adjoint L^*.

A Linear Operator without Adjoint

Let V be the vector space of polynomials over the field of complex numbers with inner product

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\, dt.$$

Define the linear operator D on V by

$$D(f) = f'.$$

Show that D has no adjoint.

A Linear Operator without Adjoint

Integration by parts shows that

$$\langle D(f), g\rangle = f(1)g(1) - f(0)g(0) - \langle f, D(g)\rangle.$$

Fix g and suppose that D has an adjoint. We must then have $\langle D(f), g\rangle = \langle f, D^*(g)\rangle$ for all f, so that

$$\begin{aligned}\langle f, D^*(g)\rangle &= f(1)g(1) - f(0)g(0) - \langle f, D(g)\rangle \\ \langle f, D^*(g) + D(g)\rangle &= f(1)g(1) - f(0)g(0).\end{aligned}$$

Since g is fixed, $L(f) = f(1)g(1) - f(0)g(0)$ is a linear functional formed as a linear combination of point evaluations.

By earlier work we know that this kind of linear functional cannot be of the the form $L(f) = \langle f, h\rangle$ unless $L = 0$.

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Since we have supposed $D^*(g)$ exists, we have for $h = D^*(g) + D(g)$ that

$$L(f) = f(1)g(1) - f(0)g(0) = \langle f, D^*(g) + D(g) \rangle = \langle f, h \rangle.$$

Since we must have $L = 0$, it follows that $g(0) = g(1) = 0$.

Hence by choosing g such that $g(0) \neq 0$ or $g(1) \neq 0$, we cannot suitably define $D^*(g)$.

The Fundamental Theorem of Linear Algebra

Theorem

Let $L : V \to V$ be a linear operator on an inner product space V. If the adjoint operator L^* exists, then

$$\mathsf{N}(L) = \mathsf{R}(L^*)^\perp \quad \text{and} \quad \mathsf{N}(L^*) = \mathsf{R}(L)^\perp.$$

Proof Idea

$$\begin{aligned}
\vec{x} \in \mathsf{N}(L) &\iff L(\vec{x}) = \vec{0} \\
&\iff \langle L(\vec{x}), \vec{y} \rangle = 0 \text{ for all } \vec{y} \in V \\
&\iff \langle \vec{x}, L^*(\vec{y}) \rangle = 0 \text{ for all } \vec{y} \in V \\
&\iff \vec{x} \perp \mathsf{R}(L^*) \\
&\iff \vec{x} \in \mathsf{R}(L^*)^\perp
\end{aligned}$$

Example

Let $V = \mathbb{R}^n$ with dot product and let $A \in \mathcal{M}_{n,n}(\mathbb{R})$.

Define the linear operator L on V by $L(\vec{x}) = A\vec{x}$.

Note that $L^*(\vec{x}) = A^*\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

The range of L^* is the column space of $A^* = A^T$,
which is the row space of A.

Thus the null space of A is the orthogonal complement of the row space of A.