**Theorem**:-

Let $T:D(T)\to Y$ be abounded linear operator with $D(T)\subseteq X$ and X,Y are normed space than the null space N(T) is closed.

**Proof**:-

Take a convergent sequence $\{X_n\}_n\in N$ in N(T)

The sequence $\{X_n\}_n \in \mathbb{N}$ is convergent so there exists some $x \in D(T)$ Such that

$\|x-x_n\| \to 0$ if $n\to\infty$

using the linearity and the boundedness of the operator T gives that

$\|T(x_n)-T(x)\| = \|T(x_n-x)\| \leq \|T\| \, \|x_n-x\| \to (n\to\infty)$

The sequence $\{X_n\}_n \in N$

$T(x_n) = 0$ for every $n\in\mathbb{N}$

The set M is closed iff every limit of sequence of M is in M.

**Theorem**:-

If x ,y are vector space and $T:x\to y$ is linear operator then:

a) R(T) the range of T is a vector space.
b) N(T) the null space of T is a vector space .

**Proof**:-

a) Take $y_1,y_2 \in R(T) \subseteq y$, then there exists $x_1,x_2 \in D(T) \subseteq X$

Such that $T(x_1)=y_1$ and $T(x_2)=y_2$

Let $\alpha\in k$ then$(y_1+\alpha y_2) \in Y$

Because Y is a vector space and

$Y \ni y_1+\alpha y_2 = T(x_1)+ \alpha T(x_2)$

$= T(x_1+\alpha x_2)$

This means that there exists an element

$Z_1=(x_1+\alpha x_2) \in D(T)$

Because D(T) is a vector space, such that

$T(z_1)= y_1+\alpha y_2$

$(y_1+\alpha y_2) \in R(T) \subseteq y.$

b) Take $x_1,x_2 \in N(T) \subseteq X$

and

Let $\alpha \in k$ then

$(x_1+\alpha x_2) \in D(T)$

and

$T(x_1+\alpha x_2) = T(x_1) + \alpha T(x_2)$

$=0$

The result$(x_1+\alpha x_2) \in N(T)$.

**Definition**:-

Let $T: x \rightarrow y$ be linear operator of x and y vector space T is invertible if there exists an operator

$S: y \rightarrow x$ such that $ST = I_x$ the identity operator on xy of $T_s = I_y$ is the identity operator on y is called the algebraic inverse of T denoted by $S = T^{-1}$

**Theorem**:-

Let x and y be vector space and T: D(T) $\rightarrow$y be linear operator with

$D(T) \subseteq x$ and $R(T) \subseteq y$ then:

1) $T^{-1} R(T) \rightarrow D(T)$ exist if and only if $T(x)=0 \Rightarrow x=0$.
2) If $T^{-1}$ exist then $T^{-1}$ is a linear operator.

**Proof**:-

1) If $T^{-1}$ exists then $\rightarrow$T is injective (why)

$\Rightarrow$T is (one-one)

Now $T(x)=0$

$T(0)=0 \Rightarrow x=0$

$\Leftarrow$ let $T(x) = T(y) \Rightarrow$ T is linear

$T(x) - T(y) = 0$

$T(x-y) = 0$ (T is linear)

$x-y = 0 \Rightarrow x=y$

$\therefore$ T is on to

$\therefore$ T is one to one

$\therefore$ T is invertible

2) The assumption that $T^{-1}$ exists the domain of $T^{-1}$ is R(T) of R(T) is vector

Let $y_1, y_2 \in R(T)$ so there exist $x_1, x_2 \in D(T)$ with $T(x_1)=y_1$, $T(x_2)=y_2$,

$T^{-1}$ exists so

$x_1=T^{-1}(y_1)$ and $x_2 = T^{-1}(y_2)$ , T is linear

$T(x_1+x_2)= T(x_1)+ T(x_2)$

$=(y_1+y_2)$

And

$T^{-1}(y_1+y_2)= x_1+x_2$

$=T^{-1}(y_1)+T^{-1}(y_2)$

$T(\alpha x_1)= \alpha y \ \exists \ T^{-1}(\alpha y_1) = \alpha x_1$

$= \alpha T^{-1}(y_1)$

$\therefore$ $T^{-1}$ is linear operator.