Problem 1

Let X be a random variable with PDF given by

$$\begin{aligned} &= \int_{-1}^{1} u^2 f_X(u)du \\ &= \tfrac{3}{2}\int_{-1}^{1} u^4 du \\ &= \tfrac{3}{5}. \end{aligned}$$

c. To find $P(X \geq \frac{1}{2})$, we can write

$$P(X \geq \frac{1}{2}) = \frac{3}{2}\int_{\frac{1}{2}}^{1} x^2 dx = \frac{7}{16}.$$

Problem 2

: 1:

Let X be a continuous random variable with PDF given by

$$f_X(x) = \frac{1}{2}e^{-|x|}, \qquad \text{for all } x \in \mathbb{R}.$$

If $Y = X^2$, find the CDF of Y.

- Solution
    - First, we note that $R_Y = [0, \infty)$. For $y \in [0, \infty)$, we have

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \tfrac{1}{2}e^{-|x|}dx \\ &= \int_{0}^{\sqrt{y}} e^{-x}dx \\ &= 1 - e^{-\sqrt{y}}. \end{aligned}$$

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Thus,

because the PDF is symmetric around $x = 0$. To find $\text{Var}(X)$, we have

$$\text{Var}(X) = EX^2 - (EX)^2 = EX^2$$

- First, note that

$$\text{Var}(Y) = \text{Var}\left(\frac{2}{X}+3\right) = 4\text{Var}\left(\frac{1}{X}\right), \qquad \text{using E}$$

Thus, it suffices to find $\text{Var}(\frac{1}{X}) = E[\frac{1}{X^2}] - (E[\frac{1}{X}])^2$. Using LOTUS, we have

$$E\left[\frac{1}{X}\right] = \int_0^1 x\left(2x+\frac{3}{2}\right)dx = \frac{17}{12}$$

$$E\left[\frac{1}{X^2}\right] = \int_0^1 \left(2x+\frac{3}{2}\right)dx = \frac{5}{2}.$$

Thus, $\text{Var}\left(\frac{1}{X}\right) = E[\frac{1}{X^2}] - (E[\frac{1}{X}])^2 = \frac{71}{144}$. So, we obtain

$$\text{Var}(Y) = 4\text{Var}\left(\frac{1}{X}\right) = \frac{71}{36}.$$

Problem 5

Let X be a positive continuous random variable. Prove that $EX = \int_0^\infty P(X \geq x)dx$.

- Solution
  - We have

$$P(X \geq x) = \int_x^\infty f_X(t)dt.$$

Thus, we need to show that

$$\int_0^\infty \int_x^\infty f_X(t)dtdx = EX.$$

1)

The discrete random variable X has the probability function

$$P(X = x) = \begin{cases} kx & x = 2, 4, 6 \\ k(x-2) & x = 8 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(a) Show that $k = \frac{1}{18}$ **(2)**

(b) Find the exact value of F(5). **(1)**

(c) Find the exact value of E(X). **(2)**

(d) Find the exact value of E(X^2). **(2)**

(e) Calculate Var($3 - 4X$) giving your answer to 3 significant figures. **(3)**

2)

A biased die with six faces is rolled. The discrete random variable X represents the score on the uppermost face. The probability distribution of X is shown in the table below.

x	1	2	3	4	5	6
$P(X = x)$	a	a	a	b	b	0.3

(a) Given that E(X) = 4.2 find the value of a and the value of b. **(5)**

(b) Show that E(X^2) = 20.4 **(1)**

(c) Find Var($5 - 3X$) **(3)**

A biased die with five faces is rolled. The discrete random variable Y represents the score which is uppermost. The cumulative distribution function of Y is shown in the table below.

y	1	2	3	4	5
F(y)	$\frac{1}{10}$	$\frac{2}{10}$	$3k$	$4k$	$5k$

(d) Find the value of k. **(1)**

(e) Find the probability distribution of Y. **(3)**

Each die is rolled once. The scores on the two dice are independent.

(f) Find the probability that the sum of the two scores equals 2 **(2)**

3)

The discrete random variable X can take only the values 1, 2 and 3. For these values the cumulative distribution function is defined by

$$F(x) = \frac{x^3 + k}{40} \qquad x = 1, 2, 3$$

(a) Show that $k = 13$

(2)

(b) Find the probability distribution of X.

(4)

Given that $\text{Var}(X) = \frac{259}{320}$

(c) find the exact value of $\text{Var}(4X - 5)$.

(2)

4)

A fair blue die has faces numbered 1, 1, 3, 3, 5 and 5. The random variable B represents the score when the blue die is rolled.

(a) Write down the probability distribution for B.

(2)

(b) State the name of this probability distribution.

(1)

(c) Write down the value of $\text{E}(B)$.

(1)

A second die is red and the random variable R represents the score when the red die is rolled.

The probability distribution of R is

r	2	4	6
$\text{P}(R = r)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

(d) Find $\text{E}(R)$.

(2)

(e) Find $\text{Var}(R)$.

(3)

5)

A discrete random variable X has the probability function

$$P(X=x)=\begin{cases} k(1-x)^2 & x=-1,0,1 \text{ and } 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $k=\frac{1}{6}$

(3)

(b) Find $E(X)$

(2)

(c) Show that $E(X^2)=\frac{4}{3}$

(2)

(d) Find $\text{Var}(1-3X)$

(3)

6)

The discrete random variable Y has probability distribution

y	1	2	3	4
$P(Y=y)$	a	b	0.3	c

where a, b and c are constants.

The cumulative distribution function $F(y)$ of Y is given in the following table

y	1	2	3	4
$F(y)$	0.1	0.5	d	1.0

where d is a constant.

(a) Find the value of a, the value of b, the value of c and the value of d.

(5)

(b) Find $P(3Y+2\geqslant 8)$.

(2)

8)

The discrete random variable X has probability distribution given by

x	-1	0	1	2	3
$P(X = x)$	$\frac{1}{5}$	a	$\frac{1}{10}$	a	$\frac{1}{5}$

where a is a constant.

(a) Find the value of a.
(2)

(b) Write down $E(X)$.
(1)

(c) Find $Var(X)$.
(3)

The random variable $Y = 6 - 2X$

(d) Find $Var(Y)$.
(2)

(e) Calculate $P(X \geqslant Y)$.
(3)

9)

The probability function of a discrete random variable X is given by

$$p(x) = kx^2 \qquad x = 1, 2, 3$$

where k is a positive constant.

(a) Show that $k = \dfrac{1}{14}$
(2)

Find

(b) $P(X \geqslant 2)$
(2)

(c) $E(X)$
(2)

(d) $Var(1 - X)$
(4)

1)

The length of time, in minutes, that a customer queues in a Post Office is a random variable, T, with probability density function

$$f(t) = \begin{cases} c(81 - t^2) & 0 \leqslant t \leqslant 9 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant.

(a) Show that the value of c is $\frac{1}{486}$ **(4)**

(b) Show that the cumulative distribution function F(t) is given by

$$F(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{6} - \frac{t^3}{1458} & 0 \leqslant t \leqslant 9 \\ 1 & t > 9 \end{cases}$$

(2)

3)

The continuous random variable X has the following probability density function

$$f(x) = \begin{cases} a + bx & 0 \leqslant x \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$$

where a and b are constants.

(a) Show that $10a + 25b = 2$ **(4)**

Given that $E(X) = \frac{35}{12}$

(b) find a second equation in a and b, **(3)**

(c) hence find the value of a and the value of b. **(3)**

(d) Find, to 3 significant figures, the median of X. **(3)**

(e) Comment on the skewness. Give a reason for your answer. **(2)**

4)

The queueing time, X minutes, of a customer at a till of a supermarket has probability density function

$$f(x) = \begin{cases} \frac{3}{32}x(k-x) & 0 \leqslant x \leqslant k \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that the value of k is 4

(4)

(b) Write down the value of E(X).

(1)

(c) Calculate Var(X).

(4)

(d) Find the probability that a randomly chosen customer's queueing time will differ from the mean by at least half a minute.

(3)

5)

The continuous random variable X has probability density function f(x) given by

$$f(x) = \begin{cases} \frac{x^2}{45} & 0 \leqslant x \leqslant 3 \\ \frac{1}{5} & 3 < x < 4 \\ \frac{1}{3} - \frac{x}{30} & 4 \leqslant x \leqslant 10 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Sketch f(x) for $0 \leqslant x \leqslant 10$

(4)

(b) Find the cumulative distribution function F(x) for all values of x.

(8)

(c) Find $P(X \leqslant 8)$.

(2)