
HW1 : Continuous Random Variables (1) – Solutions

Problem 1. Let X be a continuous random variable whose probability density function is:

$$f_X(x) = 3x^2 \mathbb{1}_{[0,1]}(x)$$

(a) Verify that f_X is a valid probability density function.

Answer. First, $f_X(x)$ is clearly nonnegative since $3x^2 \geq 0$ for $0 \leq x \leq 1$ and 0 otherwise. Second, let us show that $\int_{\mathbb{R}} f_X = 1$:

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_0^1 3x^2 dx = x^3 \Big|_0^1 = 1$$

(b) What is $\mathbb{P}(X = 1/2)$?

Answer. X is a continuous random variable. Hence, the probability that X takes on a particular value is zero : $\mathbb{P}(X = 1/2) = 0$.

(c) What is $\mathbb{P}(1/2 < X < 1)$?

Answer.

$$\begin{aligned} \mathbb{P}(1/2 < X < 1) &= \int_{1/2}^1 f_X(x) dx \\ &= \int_{1/2}^1 3x^2 dx \\ &= x^3 \Big|_{1/2}^1 \\ &= 1 - \frac{1}{8} \\ &= \frac{7}{8} \end{aligned}$$

(d) Give the cumulative distribution function F_X of X .

Answer. The cumulative distribution function F_X of X is piecewise like its probability density function :

- If $x < 0$, then $f_X(t) = 0$ for all $t \leq x$.

$$F_X(x) = \int_{-\infty}^x f_X(t) dt = \int_{-\infty}^x 0 dt = 0$$

- If $0 \leq x \leq 1$, then $f_X(t) = 3t^2$ for all $0 \leq t \leq x$.

$$\begin{aligned} F_X(x) &= F_X(0) + \int_0^x f_X(t) dt \\ &= 0 + \int_0^x 3t^2 dt \\ &= t^3 \Big|_0^x \\ &= x^3 \end{aligned}$$

- If $x > 1$, then $f_X(t) = 0$ for all $1 \leq t \leq x$.

$$\begin{aligned} F_X(x) &= F_X(1) + \int_1^x f_X(t) dt \\ &= 1^3 + 0 \\ &= 1 \end{aligned}$$

In a nutshell, F_X is given by :

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^3 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

- (e) What is the expected value of X ?

Answer.

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= \int_0^1 3x^3 dx \\ &= \frac{3}{4} x^4 \Big|_0^1 \\ &= \frac{3}{4} \end{aligned}$$

- (f) What is the variance of X ?

Answer. We can use the shortcut formula : $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$. Since the expectation was previously computed, we only need to calculate $\mathbb{E}[X^2]$:

$$\begin{aligned}\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx \\ &= \int_0^1 3x^4 dx \\ &= \frac{3}{5} x^5 \Big|_0^1 \\ &= \frac{3}{5}\end{aligned}$$

Therefore, we obtain :

$$\text{Var}(X) = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{80}$$

Problem 2. Let X be a continuous random variable whose probability density function is:

$$f_X(x) = \frac{x^3}{4} \mathbf{1}_{[0,c]}(x)$$

What is the value of the interval's upper bound c that makes f_X a valid probability density function?

Answer. f_X is a valid probability density function if the following holds :

$$\begin{aligned}\int_{-\infty}^{\infty} f_X(x) dx = 1 &\Leftrightarrow \int_0^c \frac{x^3}{4} dx \\ &\Leftrightarrow \frac{x^4}{16} \Big|_0^c = 1 \\ &\Leftrightarrow c^4 = 16 \\ &\Leftrightarrow c = 2 \text{ or } c = -2\end{aligned}$$

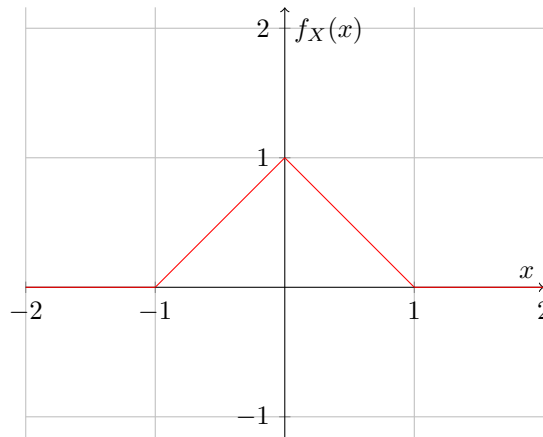
We discard the negative solution since $[0, c]$ is an interval, thus $c \geq 0$. So the value of the upper bound is $c = 2$.

Problem 3. Let X be a continuous random variable whose probability density function is:

$$f_X(x) = \begin{cases} 0 & x < -1 \\ x + 1 & -1 \leq x \leq 0 \\ 1 - x & 0 < x \leq 1 \\ 0 & x > 1 \end{cases}$$

(a) Graph the probability density function f_X of X .

Answer.



(b) Verify that f_X is a valid probability density function.

Answer. It is obvious that f_X is nonnegative. Let us show that $\int_{\mathbb{R}} f_X = 1$:

$$\begin{aligned}
 \int_{-\infty}^{\infty} f_X(x) dx &= \int_{-\infty}^{-1} f_X(x) dx + \int_{-1}^0 f_X(x) dx + \int_0^1 f_X(x) dx + \int_1^{\infty} f_X(x) dx \\
 &= 0 + \int_{-1}^0 (x+1) dx + \int_0^1 (1-x) dx + 0 \\
 &= \left[\frac{x^2}{2} + x \right]_{-1}^0 + \left[x - \frac{x^2}{2} \right]_0^1 \\
 &= \left\{ 0 - \left(\frac{1}{2} - 1 \right) \right\} + \left\{ \left(1 - \frac{1}{2} \right) - 0 \right\} \\
 &= 1
 \end{aligned}$$

(c) Give the cumulative distribution function F_X of X .

Answer. The cumulative distribution function F_X of X is piecewise like its probability density function :

- If $x < -1$, then $f_X(t) = 0$ for all $t \leq x$.

$$F_X(x) = \int_{-\infty}^x f_X(t) dt = \int_{-\infty}^x 0 dt = 0$$

- If $-1 \leq x \leq 0$, then $f_X(t) = t + 1$ for all $-1 \leq t \leq x$.

$$\begin{aligned}
 F_X(x) &= F_X(-1) + \int_{-1}^x f_X(t) dt \\
 &= 0 + \int_{-1}^x (t + 1) dt \\
 &= \left. \frac{t^2}{2} + t \right|_{-1}^x \\
 &= \frac{x^2}{2} + x - \left(\frac{1}{2} - 1 \right) \\
 &= \frac{x^2}{2} + x + \frac{1}{2} \\
 &= \frac{1}{2}(x + 1)^2
 \end{aligned}$$

- If $0 < x \leq 1$, then $f_X(t) = 1 - t$ for all $0 < t \leq x$.

$$\begin{aligned}
 F_X(x) &= F_X(0) + \int_0^x f_X(t) dt \\
 &= \frac{1}{2}(0 + 1)^2 + \int_0^x (1 - t) dt \\
 &= \frac{1}{2} + \left[t - \frac{t^2}{2} \right]_0^x \\
 &= \frac{1}{2} + \left\{ \left(x - \frac{x^2}{2} \right) - 0 \right\} \\
 &= -\frac{x^2}{2} + x + \frac{1}{2}
 \end{aligned}$$

- If $x > 1$, then $f_X(t) = 0$ for all $1 \leq t \leq x$.

$$\begin{aligned}
 F_X(x) &= F_X(1) + \int_1^x f_X(t) dt \\
 &= \left(-\frac{1^2}{2} + 1 + \frac{1}{2} \right) + 0 \\
 &= 1
 \end{aligned}$$

In a nutshell, F_X is given by :

$$F_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{1}{2}(x + 1)^2 & \text{if } -1 \leq x \leq 0 \\ -\frac{x^2}{2} + x + \frac{1}{2} & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

(d) Graph the cumulative distribution function F_X of X .

Answer.

