الجامعة المستنصرية
كلية التربية
قسم الفيزياء
المرحلة: الأولى
الدراسة الصباحية
المادة: الرياضيات

Section Twenty

LOGARITHM AND EXPONENTIAL FUNCTION INTEGRALS

1. The Integral of Logarithm Function

If u is a differentiable function that is never zero, then

$$\int \frac{1}{u} du = \ln|u| + c$$

In the sense (بمعنى) that the derivative of the numerator is equal the denominator, the integration is the denominator, ln.

EXAMPLES

1. $\int \frac{dx}{x} = \ln|x| + c$
2. $\int \frac{2xdx}{x^2+1} = \ln|x^2 + 1| + c$
3. $\int \frac{(x+1)dx}{x^2+2x+3} = \frac{1}{2}\int \frac{(2x+2)dx}{x^2+2x+3} = \frac{1}{2}\ln|x^2 + 2x + 3| + c$
4. $\int \frac{x^2dx}{1-x^3} = \frac{3}{3}\int \frac{x^2dx}{1-x^3} = \frac{-1}{3}\int \frac{-x^2}{1-x^3} = \frac{-1}{3}\ln|1 - x^3| + c$
5. $\int \frac{\sin x dx}{1-\cos x} = -\ln|1\text{-}\cos x|+c$

$6.\int \frac{\sec^2 x dx}{3+\tan x} = \ln|3+\tan x|+c$

$7.\int \frac{(x-2)dx}{x^2-4x-3} = \frac{1}{2}\int \frac{(2x-4)dx}{x^2-4x-3} = \frac{1}{2}\ln|x^2-4x-3|+c$

$8.\int \frac{\cos x dx}{1+\sin x} = \ln|1+\sin x|+c$

9.Prove $\int \frac{\sin 2x dx}{\sin^2 x} = \ln|\sin^2 x|+c$

Solution: Let $u = \sin^2 x$

$du = 2\sin x\cos x dx$

$\sin 2x = 2 \text{si} x \cos x$

$\text{si} x \cos x = \frac{1}{2}\sin(x-x) + \frac{1}{2}\sin(x+x) = \frac{1}{2}\sin 2x$

$du = \frac{1}{2} 2\sin 2x = \sin 2x$

$\int \frac{du}{u} = \ln|u|+c = \ln|\sin^2 x|+c$

EVALUATE

1. $\int \frac{\ln x}{x} dx = \int (\ln x) \frac{1}{2} dx = \frac{(\ln x)^2}{2} + c$

2. $\int \frac{(\ln x)^3}{x} dx = \int (\ln x)^3 \frac{1}{x} dx = \frac{(\ln x)^4}{4} + c$

3. $\int \frac{dx}{x \ln x} = \int \frac{(x)^{-1}}{\ln x} dx = \ln|\ln x| + c$

4. $\int \frac{\sqrt{1-\ln x}}{x} dx = \int (1 - \ln x)^{1/2} \frac{1}{x} dx = -\frac{(1-\ln x)^{3/2}}{3/2} + c$

The Integrals of tanx, cotx, secx, and cscx

1. $\int \text{tanx}dx = \int \frac{\text{sinx}}{\text{cosx}}dx = \int \frac{-du}{u}$

$$u = \text{cosx} > 0 \text{ on } (-\pi/2, \pi/2)$$

$$du = -\sin x\, dx$$

$$= -\ln|x| + c = -\ln|\text{cosx}| + c$$

$= \ln \frac{1}{|\text{cosx}|} + c = \ln|\text{secx}| + c$ Reciprocal Rule

2. $\int \text{cotx}dx = \int \frac{\text{cosx}}{\text{sinx}}dx = \int \frac{du}{u}$

$$u = \text{sinx}$$

$$du = \text{cosx}\, x\, dx$$

$$= \ln|x| + c = \ln|\text{sinx}| + c$$

$= \ln \frac{1}{\text{sinx}} + c = -\ln|\text{cscx}| + c$ Reciprocal Rule

To integrate secx, we multiply and divide by (sec x + tanx).

3. $\int \sec x dx = \int \sec x \frac{\sec x+\tan x}{\sec x+\tan x} dx = \int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x} dx$

$$u = \sec x + \tan x$$

$$du = (\sec^2 x + \sec x\tan x)dx$$

$$= \int \frac{du}{u} = c = \ln|u| + c = c = \ln|\sec x + \tan x| + c$$

$$= \ln|\sec x + \tan x| + c$$

For cscx, we multiply and divide by (cscx + cotx).

4. $\int \csc x dx = \int \csc x \frac{\csc x+\cot x}{\csc x+\cot x} dx = \int \frac{\csc^2 x+\csc x\cot x}{\csc x+\cot x} dx$

$$= -\int \frac{-\csc^2 x-\csc x\cot x}{\csc x+\cot x} dx$$

$$u = \csc x + \cot x$$

$$du = (-\csc x\cot x - \csc^2 x)dx$$

$$= -\ln|\csc x + \cot x| + c$$

The Integrals of $\log_a x$

To find integrals involving base a logarithms, we convert them to natural logarithms. If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{1}{u}\frac{du}{dx}$$

$$\int \log_a x du = \frac{1}{\ln a}\int \ln u du$$

EXAMPLE: Find $\int \frac{\log_2 x}{x} dx$

Solution: $\int \frac{\log_2 x}{x} dx = \frac{1}{\ln 2}\int \frac{\ln x}{x} dx = \int u du = \frac{u^2}{2} + c$

$$\log_2 x = \frac{\ln x}{\ln 2}, \qquad u = \ln x \qquad du = \frac{1}{x} dx$$

$$\int \frac{\log_2 x}{x} dx = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + c = \frac{(\ln x)^2}{2\ln 2} + c$$

.

2. The Integral of Exponential Function

The general antiderivative of the exponential function

1. If $y = e^u$

$$y' = e^u \frac{du}{dx}$$

$$\int e^u \, du = e^u + c$$

2. If $y = a^u$

$$y' = a^u \ln a \frac{du}{dx}$$

$$\int a^u du = \frac{a^u}{\ln a} + c$$

EXAMPELS

1. $\int \frac{e^x}{1+3e^x}dx = \frac{1}{3}\int 3e^x(1+3e^x)^{-1}dx = \frac{1}{3}\int \frac{3e^x}{(1+3e^x)}dx$

$= \frac{1}{3}\ln|1+3e^x| + c$

2. $\int 3x^3e^{-2x^4}dx = -\frac{3}{8}\int -8x^3e^{-2x^4}dx = -\frac{3}{8}e^{-2x^4}+c$

3. $\int 2^{-4x}dx = -\frac{1}{4}\int 2^{-4x}(-4)dx = -\frac{1}{4}2^{-4x}\frac{1}{\ln 2} + c$

4. $\int \frac{e^{\tan x}}{\cos^2 x}dx = \int e^{\tan x}\sec^2 x dx = e^{\tan x} + c$

5. $\int \frac{e^x}{1+e^x}dx = \ln|1+e^x| + c$

6. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx = \ln|e^x+e^{-x}| + c$

7. $\int \frac{dx}{1+e^x} = \int \frac{1}{1+e^x}\frac{e^{-x}}{e^{-x}}dx = \int \frac{e^{-x}}{1+e^{-x}}dx = \ln|1+e^x| + c$

8. $\int (\frac{e^{-2x}-1}{e^{-2x}+1})dx = \int (\frac{e^{-2x}-1}{e^{-2x}+1})\frac{e^{-x}}{e^{-x}}dx = \int \frac{e^x+e^{-x}}{e^x+e^{-x}}dx$

$= \ln|e^x+e^{-x}| + c$