Chapter 4 The special theory of relativity

The special theory of relativity

Before the beginning of the twentieth century , two branches of physics (mechanics and electromagnetism) had developed quite independently of each other , the laws of mechanics and electromagnetism had been verified to such an extent that physicists were sure that there would be no further modification or improvements of them, but to every one's surprise ,early in this century ,physicist were faced with many new and basic problems, first they found that Newton's second law of motion which had been so well established for objects moving with low speeds ,did not give the correct results when applied to object moving with high speeds (speeds comparable to the speed of light) . Second, they found that for two observers in relative motion, one could not use the same set of transformation equations to transform the laws of mechanics and electromagnetism from the frame of reference of one observer to the frame of reference of the other observer. These and other difficulties were overcome by the formulation of the special theory of relativity by Einstein in 1905.

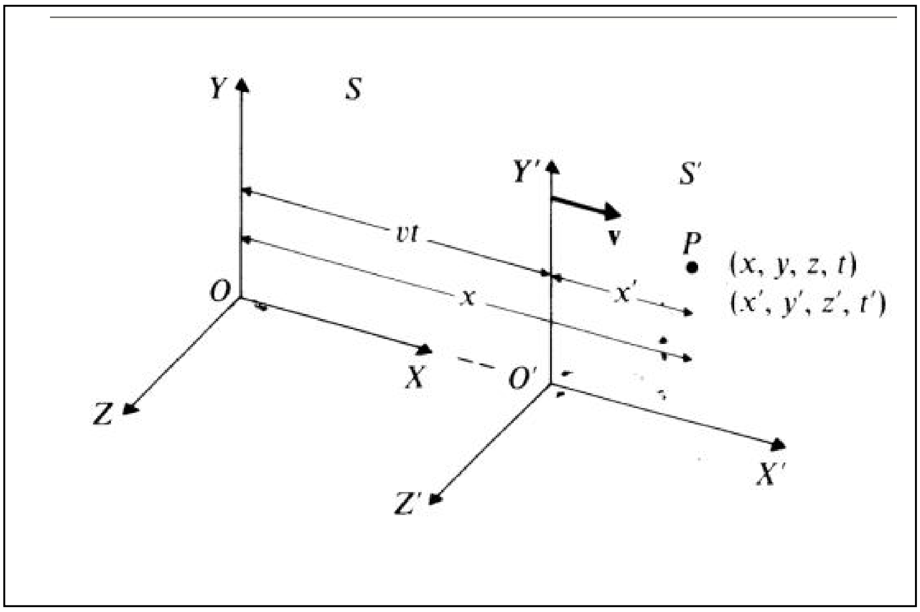
Galilean transformation

According to Newton first law, a system at rest will remain at rest or a system in uniform motion will remain in uniform motion if no net external forces act on the system.

Such systems in which the law of inertia holds are called inertial systems. For all practical purpose, a set of coordinates axes attached to the earth may be regarded as an inertial system, provided we neglect the small acceleration resulting from the rotational and orbital motion of the earth.

Let us see how we transfer the coordinates of an event from one inertial system to another inertial system which is moving with a uniform velocity relative to the first. In Newtonian mechanics such transformation are made by the Galilean transformation equations.

Let one set of coordinates axis xyz be located in an inertial system S ,set x'y'z' in an inertial system which is moving with respect to system S with a velocity v along the xx' axes as shown in fig(1).



The origins of the two inertial systems coincide at t=t'=0. let the coordinates of an event taking place at some point P be (x,y,z,t) and (x', y', z', t') in the inertial systems respectively . According to fig (1). These coordinate are related by the Galilean coordinate transformation equations

$$\begin{aligned} x' &= x - vt & \qquad & x = x' + vt \\ y' &= y & & y = y' \\ z' &= z & \text{or} \qquad & z = z' \qquad \ldots\ldots\ldots(1) \\ t' &= t & & t = t' \end{aligned}$$

Note : that in Newtonian relativity we always assume that $t' = t$

If we differentiate the above equations, assuming that $\frac{d}{dt}$ and $\frac{d}{dt'}$ are identical we get the following velocity transformation equations

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v \qquad u'_x = u_x - v$$

$$\frac{dy'}{dt'} = \frac{dy}{dt} \qquad \text{That is} \qquad u'_y = u_y \qquad \ldots\ldots(2)$$

$$\frac{dz'}{dt'} = \frac{dz}{dt} \qquad u'_z = u_z$$

Differentiating once again, we get acceleration transformation equations

$$\frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2} \qquad a'_x = a_x$$

$$\frac{d^2y'}{dt'^2} = \frac{d^2y}{dt^2} \qquad \text{That is} \qquad a'_y = a_y \qquad \ldots\ldots\ldots(3)$$

$$\frac{d^2z'}{dt'^2} = \frac{d^2z}{dt^2} \qquad a'_z = a_z$$

That is, the acceleration is the same as viewed from either inertial system. We can go step further and show that an equation which describes an event in reference frame does not change its form when transferred to another reference by using Galilean transformation for example. The components of a force F acting on a particle of mass m at a point P in reference system S may be written as

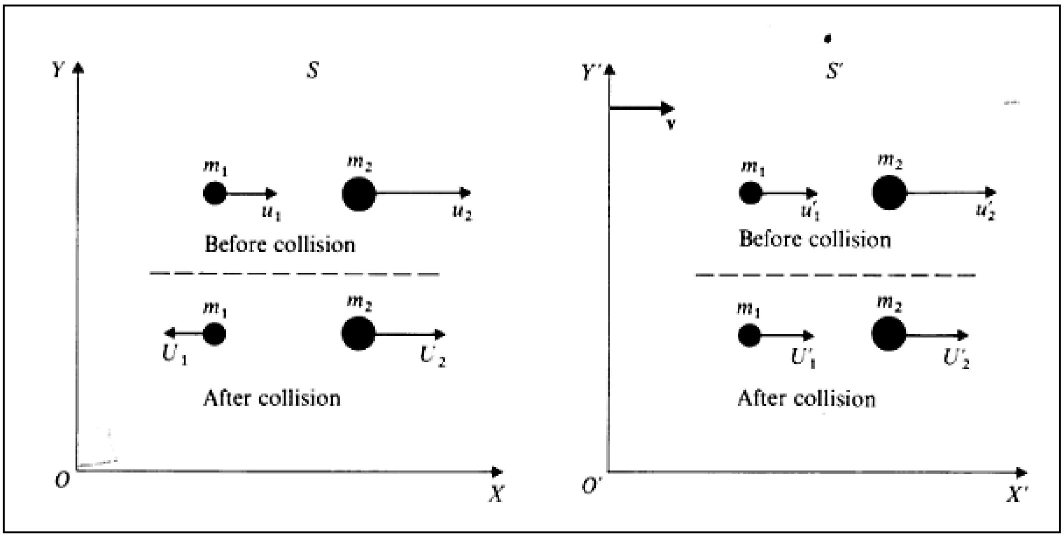
$$F_x = m\frac{d^2x}{dt^2} \quad , \quad F_y = m\frac{d^2y}{dt^2} \quad , \quad F_z = m\frac{d^2z}{dt^2}$$

Using the values of xyz and t or directly from equation (3) we get

$$F'_x = m\frac{d^2x'}{dt'^2} \quad , \quad F'_y = m\frac{d^2y'}{dt'^2} \quad , \quad F'_z = m\frac{d^2z'}{dt'^2}$$

And this implies that the form of the equation has not changed under Galilean transformation (Newton's second law)

Example (1) illustrates the invariance of the form of the conservation of linear momentum and kinetic energy under Galilean transformation.



Consider a collision between two masses, the conservation of linear momentum and kinetic energy may be written as

$$m_1u_1 + m_2u_2 = m_1U_1 + m_2U_2$$

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1U_1^2 + \frac{1}{2}m_2U_2^2$$

Let us observe this collision from another inertial system S' moving with velocities u_1' and u_2' a long the x' -axis

$$m_1(u_1^{'} + v) + m_2(u_2^{'} + v) = m_1(U_1^{'} + v) + m_2(U_2^{'} + v)$$

This is on simplification

$$m_1u_1^{'} + m_2u_2^{'} = m_1U_1^{'} + m_2U_2^{'}$$

Now

$$\frac{1}{2}m_1(u_1^{'} + v)^2 + \frac{1}{2}m_2(u_2^{'} + v)^2 = \frac{1}{2}m_1(U_1^{'} + v)^2 + \frac{1}{2}m_2(U_2^{'} + v)^2$$

By simplification we get

$$\frac{1}{2}m_1u_1^{'2} + \frac{1}{2}m_2u_2^{'2} = \frac{1}{2}m_1U_1^{'2} + \frac{1}{2}m_2U_2^{'2}$$

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Question

Does Galilean transformation hold good for electromagnetism

Answer

Let us take an example a spherical electromagnetic wave propagating with a constant speed c in the reference system S may be given by

$$x^2 + y^2 + z^2 - c^2t^2 = 0$$

For this equation to be invariant, its form in the system S' should be

$$x'^2 + y'^2 + z'^2 - c^2t^2 = 0$$

Substituting x,y,z, and t from equation (1) we get

$$(x' + vt) + y'^2 + z'^2 - c_2t^2 = 0$$

Thus Galilean transformation hold well for classical but not for electromagnetism.

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Einstein's Postulates

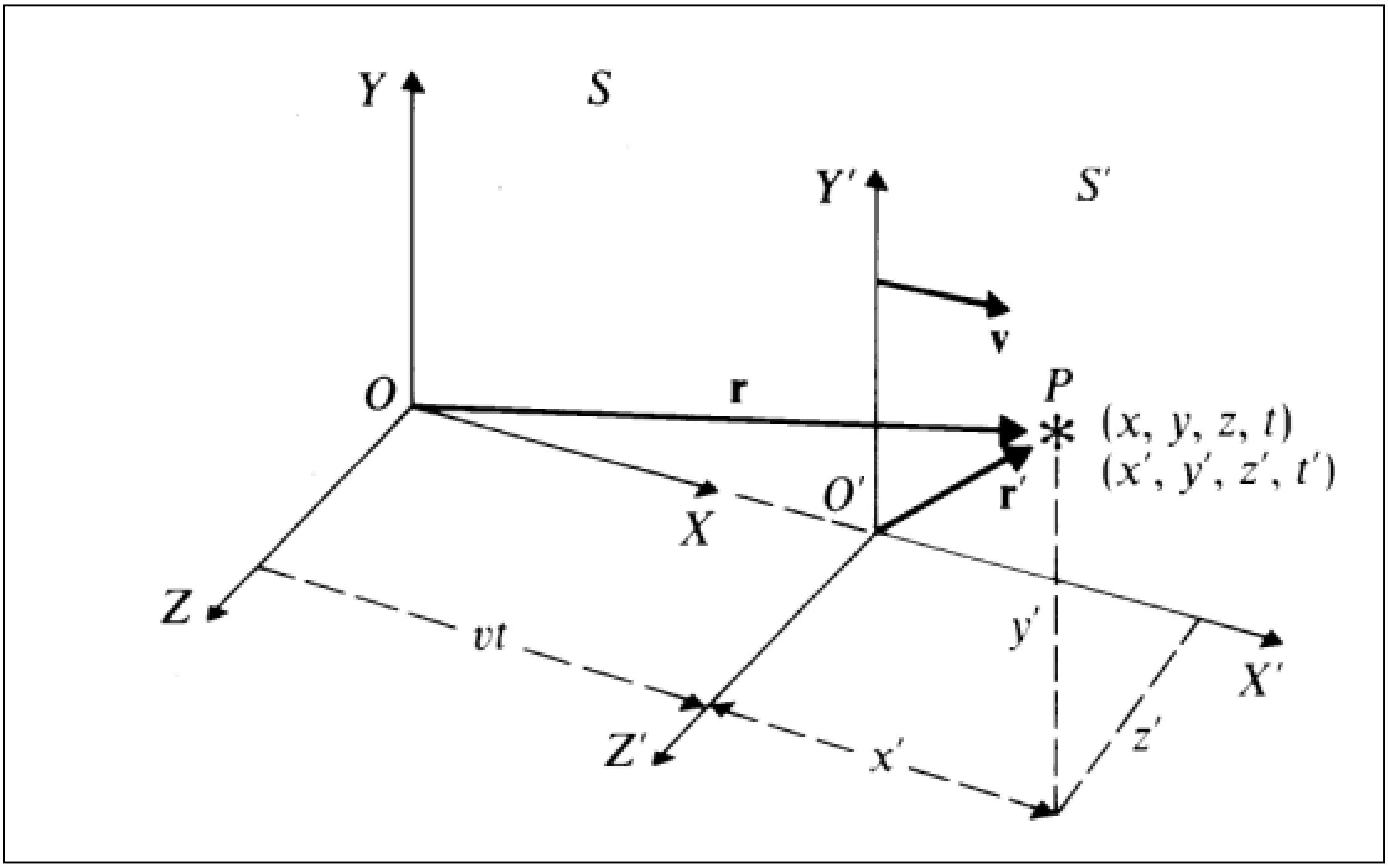
Postulate (1) the principle of equivalence (or relativity).

The laws of physics are the same in all inertial frames that is all inertial frames are equivalent.

Postulate (2) the principle of the constancy of the speed of light. The speed of light in free space (vacuum) is always a constant equal to C and is independent of the relative motion of the inertial systems, the source and the observer.

Lorentz coordinate transformation

Two inertial systems one at rest and the other moving with a uniform velocity v .there are two observers one at rest with respect to S and the other at rest with respect to S'. The coordinates of the two inertial systems coincide at t=t'=0. At the same moment a flash of light is emitted from the common origin of S and S' after a certain time the flash point reaches point P



Since the relative motion of the system S and S' is along the xx' axis

$\therefore y = y' \quad z = z'$

$x' = x'(x,t)$

$t' = t'(t,x)$

If we assume that the transformation is linear

$x' = A_1 x + B_1 t .....(4)$

$t' = A_2 t + B_2 x ......(5)$

Let us consider the motion of the origin of the system S' for which

x'=0 at t'=0

After a time t we find that according to S, the system appears to be moving with a velocity v so that $x = vt$

$\therefore B_1 = -vA_1$

Hence $x' = A_1(x - vt)$

Substitute in equation (2) we get

$$(A_1^2 - c^2 B_2^2)x^2 + y^2 + z^2 - 2(A_1^2 v + c^2 A_2 B_2)xt + (A_1^2 v^2 - c^2 A_2^2)t^2 = 0$$

Hence we get

$A_1^2 - c^2 B_2^2 = 1$

$A_1^2 v + c^2 A_2 B_2 = 0$

$-A_1^2 v^2 + c^2 A_2^2 = c^2$

By solving these equation

$$A_1 = A_2 = \frac{1}{\sqrt{1 - v^2 / c^2}} = \gamma$$

$$B_2 = -\gamma \frac{v}{c^2}$$

Thus using these value in equation (4) and (5) we get

$x' = \gamma(x - vt)$

$y' = y$

$z' = z$ Lorentz coordinates system equation from system S to S' ..(6)

$t' = \gamma(t - (v/c^2)x)$

We obtain the inverse transformation from to S by replacing v by –v

$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma(t' - (v/c^2)x')$$

Lorentz coordinates system equation from system S' to S ..(7)

When the situation involves very low velocities, we can see that in case of $v \to 0, \frac{v}{c} << 1, \frac{v^2}{c^2} <<< 1$ thus equation (7) reduce to

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

That is

Limit (Lorentz Transformation)=Galilean transformation

$v \to 0$

Lorenz velocity transformation

Let us consider two inertial systems S and S' moving with a relative velocity v along the x x' axis .Consider a particle at P , which is moving in space and has velocity $u(u_x, u_y, u_z)$ as measured by an observer in system S and velocity u' (u'_x, u'_y, u'_z) as measured by an observer in system S' . The aim is to find relations between the components

$$u_x = \frac{dx}{dt} \quad , \quad u_y = \frac{dy}{dt} \quad , \quad u_z = \frac{dz}{dt}$$

And the component

$$u'_x = \frac{dx'}{dt'} \qquad u'_y = \frac{dy'}{dt'} \qquad u'_z = \frac{dz'}{dt'}$$

Differentiating equation (6) we obtain

$$dx' = \gamma(dx - vdt)$$
$$dy' = dy$$
$$dz' = dz$$
$$dt' = \gamma(dt - (v/c^2)dx)$$

Thus the component of u' are given by

$$u'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - (vdx/c^2)} = \frac{(dx/dt) - v}{1 - (v/c^2)(dx/dt)}$$

Or

$$u'_x = \frac{u_x - v}{1 - (vu_x/c^2)}$$

$$u'_y = \frac{u_y}{\gamma\left[1 - (vu_x/c^2)\right]} \quad \text{Lorentz velocity transformation} \quad ....(8)$$

$$u'_z = \frac{u_z}{\gamma\left[1 - (vu_x/c^2)\right]}$$

Inverse transformation

$$u_x = \frac{u'_x + v}{1 + (vu'_x/c^2)}$$

$$u_y = \frac{u'_y}{\gamma\left[1 + (vu'_x/c^2)\right]} \quad \text{Lorentz velocity transformation} \quad ....(9)$$

$$u_z = \frac{u'_z}{\gamma\left[1 + (vu'_x/c^2)\right]}$$

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Length contraction

Consider two observers at rest in inertial system S and S'. The observer in S' has a rod of length L_o lying parallel to the x'axis. That is $L_o = x'_2 - x'_1$ is the same for both observers when they are at rest relative to each other. Now let us assume that the S' system starts moving with velocity v along the xx' -axis. To the observer in S', the length of the rod is still L_o, but the observer in S the length of the rod is $L = x_2 - x_1$, where we must find x_2 and x_1 by making use of the Lorentz transformation that is

$$x'_2 = \gamma(x_2 - vt_2) \qquad\qquad x'_1 = \gamma(x_1 - vt_1)$$

On subtraction, this gives

$$x'_2 - x'_1 = \gamma(x_2 - x_1) - v(t_2 - t_1)$$

Since the observer in the S system must measure the two ends of the rod simultaneously it means that $t_1 = t_2$ so

$$x_2 - x_1 = \frac{1}{\gamma}(x'_2 - x'_1)$$

$$L_o = L\gamma$$

$$L = L_o\sqrt{(1 - v^2/c^2)}$$

From this equation we find

$1 - L \approx L_o \qquad 0 \le v \le 0.01c$

$2 - L = 0$ When $\quad v = c$

$3 - L < L_o \qquad 0.01c < v < c$

The effect is reciprocal. If S had a rod of length L_o, while S' is moving and looks at the rod, it will appear constructed to him, that is

$$L' = L_o\sqrt{1 - \frac{v^2}{c^2}}$$

Thus the measured length of an object is maximum when the object is at rest relative to the observer and appears constructed by factor $\sqrt{1-\beta^2}$ to an observer who is in motion relative to the object.

Example

frame S' has a speed v=0.6c relative to S .clocks are adjusted so that t=t'=0 at x=x'=0,two events occur, event(1) occurs at x_1=10 m ,t_1=2x10^{-7} s , event(2) occur at x_2=50 m,t_2=3x10^{-7}s .what is the distance between the events as measured in S'?

Solution

$$\frac{v^2}{c^2}=\frac{9}{25}$$

Hence $\gamma=\sqrt{1-\beta^2}$ =5/4

$$x_2'-x_1'=\frac{5}{4}(50-10)-\frac{3}{5}(3x10^8)(3-2)x10^7$$

=27.5 m

Dilation of time

A time interval, like a length interval is not absolute. Consider two different events. The time intervals between these two events are registered on two different clocks, in inertial systems S and S' both clocks are observed from system

$$t_1=\frac{t_1'+(v/c^2)x_1'}{\sqrt{1-\beta^2}}$$

$$t_2=\frac{t_2'+(v/c^2)x_2'}{\sqrt{1-\beta^2}}$$

$$t_2-t_1=\frac{(t_2'-t_1')+(v/c^2)(x_2'-x_1')}{\sqrt{1-\beta^2}}$$

According to this equation, it is the observer in system S who looks at the clock of S' which must be at rest during the observation (that is $x_1'=x_2'$), reads a time interval of $t_2'-t_1'$ and then compares it with his own interval ($t_2=t_1$) thus

$$t_2 - t_1 = \frac{(t_2' - t_1')}{\sqrt{1-\beta^2}}$$

If we define t_2'-t_1' as the interval between the two events as the interval as recorded by an observer in S who is in motion (with velocity –v with respect to S' and its clock), we obtain the following relation

$$T = \frac{T_o}{\sqrt{1-\beta^2}}$$

$T = T_o \qquad 0 \le v \le 0.01c$

$T = \infty \qquad v = c$

$T > T_o \qquad 0.01c < v < c$

Thus a clock appears to run as it's fastest when it is at rest relative to the observer and its rate seems to be slowed down by a factor of $\sqrt{1-\beta^2}$ when the clock is moving with a velocity v relative to the observer.

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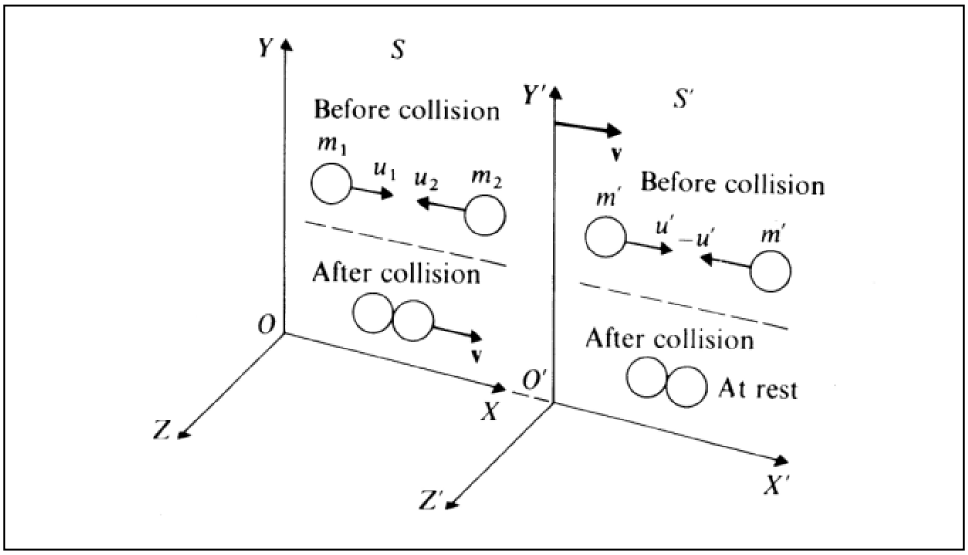
Mass and momentum in relativity

Consider a perfectly inelastic collision between two balls in the inertial system S' which is moving relative to system S with velocity v along the



xx' axis the conservation of linear momentum as applied to the inertial systems S and S'

$m'u' + m'(-u') = 0$ in S'……(1)

$m_1u_1 + m_2u_2 = (m_1 + m_2)v$ in S ….(2)

If we let $M = m_1 + m_2$ …(3)

We may write equation (2) as

$m_1u_1 + m_2u_2 = Mv$ ….(4)

Now subtract m_1u_2 , m_2u_1 separately from equation (4) and rewrite we will get

$m_1(u_1 - u_2) = M(v - u_2)$ ….(5)

$m_2(u_1 - u_2) = M(u_1 - v)$ …(6)

Where according to the Lorentz velocity transformation

$$u_1 = \frac{u' + v}{1 + (uv/c^2)} \quad , \quad u_2 = \frac{-u' + v}{1 - (u'v/c^2)} \quad \text{….(7)}$$

Dividing equation (5) ,(6) one by another and substituting for $u_1$and u_2 we get

$$\frac{m_1}{m_2}=\frac{v-u_2}{u_1-v}=\frac{1+(u'v/c^2)}{1-(uv/c^2)} \quad .....(8)$$

Using the value of u_1 and u_2 from equation (7) we get

$$1+\frac{u'v}{c^2}=\frac{\sqrt{(1-v^2/c^2)(1-u'^2/c^2)}}{\sqrt{(1-u_1^2/c^2)}} \quad ....(9)$$

Similarly

$$1-\frac{u'v}{c^2}=\frac{\sqrt{(1-v^2/c^2)(1-u'^2/c^2)}}{\sqrt{(1-u_2^2/c^2)}} \quad ..(10)$$

Substituting (10) and (9) into (8) we get

$$\frac{m_1}{m_2}=\frac{\sqrt{(1-u_2^2/c^2)}}{\sqrt{(1-u_1^2/c^2)}} \quad ....(11)$$

Or

$$m_1(\sqrt{1-u_1^2/c^2})=m_2(\sqrt{1-u_2^2/c^2})=constant=m_o \quad ....(12)$$

Where m_o is the rest mass of the identical balls.

Thus in general if a mass s moving with a velocity v relative to an observer

$$m\sqrt{1-v^2/c^2}=m_o$$

Or $m=\gamma m_o$ .....(13)

This equation states that the mass of an object is not constant in general only when the object is at rest the mass will be constant and equal to m_o when the object starts to move ,its mass appears to be increased to m this is called relativistic mass.

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Example

At what speed will the mass of a proton become double its rest mass?

Solution

$$m = \gamma m_o$$

$$m = 2m_o$$

$$2m_o = \gamma m_o$$

$$\therefore \gamma = 2$$

$$= \frac{1}{1 - v^{2/c^2}} = 4 \Rightarrow 1 = 4 - 4v^2/c^2 = 3/4$$

$$v = \frac{\sqrt{3}}{2}c$$

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Relativistic Mechanics

The linear momentum of a body is given by

$$P = mv$$

$$P = \gamma m_o v$$

In relativistic mechanics, the force acting on the body is defined as the time rate of change of relativistic momentum

$$F = \frac{dP}{dt} = \frac{d}{dt}(mv) = \frac{d}{dt}(\gamma m_o v)$$

Since m is no longer constant

$$\frac{d(mv)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

Suppose that a particle of rest mass m_o is acted on by a force F through a distance x in time t along the x-axis and that it attains a final velocity v. the kinetic energy k mat be defined as the work done by the force F

$$k = \int_0^x F dx = \int_0^x \frac{d}{dt}(mv)dx = \int_0^v \frac{dx}{dt} d(\gamma m_o v) = m_o \int_0^v v d(\gamma v) \quad \ldots.(1)$$

Note that

$$d\gamma = d(1 - \frac{v^2}{c^2})^{-1/2} = \frac{v}{c^2}(1 - \frac{v^2}{c^2})^{-3/2} dv$$

And that

$$d(\gamma v) = \gamma dv + v d\gamma$$

$$= \left[(1 - \frac{v^2}{c^2})^{-1/2} dv + \frac{v^2}{c^2}(1 - \frac{v^2}{c^2})^{-3/2} dv = \frac{dv}{(1 - v^2/c^2)^{3/2}} \ldots\ldots.(2) \right]$$

Substituting for $d\gamma v$ in equation (1) and integrating we obtain

$$k = m_o \int_0^v v d(\gamma v) = m_o \int_0^v \frac{v dv}{(1 - v^2/c^2)^{3/2}} = m_o c^2 \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right] = \frac{m_o c^2}{\sqrt{1 - v^2/c^2}} - m_o c^2$$

That is

$$k = mc^2 - m_o c^2 = (\gamma - 1) m_o c^2$$

Where mc^2 is the total energy E and $m_o c^2$ is the rest mass energy E_o that is

$k = E - E_o$

The relation between momentum and energy in relativity

There is a simple and very useful relation between relativistic momentum P,rest mass energy E_o and the total energy E which can obtains as follows

$$m = \frac{m_o}{\sqrt{1 - v^2 / c^2}}$$

Squaring and rearranging we get

$$m^2(1 - v^2 / c^2) = m_o$$

Multiply both side by c^4 ,we get

$$m^2c^4 - m^2v^2c^2 = m_o^2c^4$$

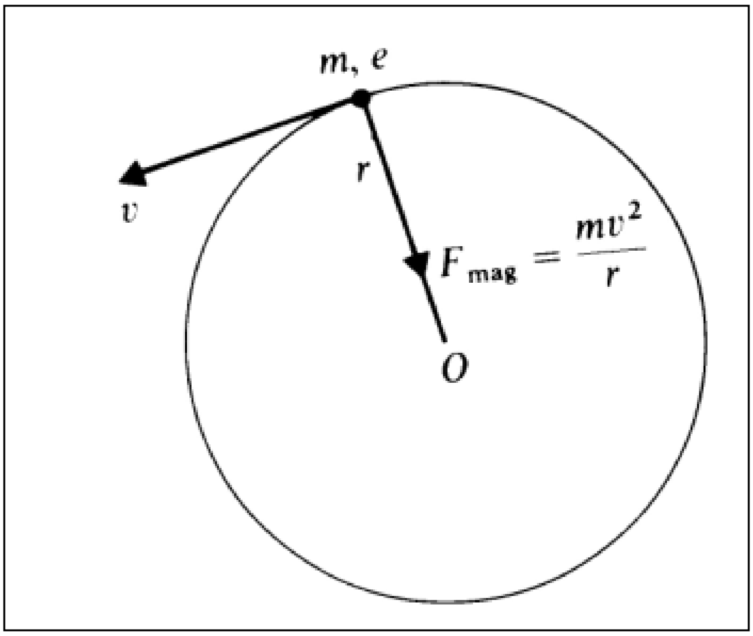
Substituting for p=mv , $E_o=m_oc^2$, and $E=mc^2$ we obtain the required relation

$$E^2 = P^2c^2 + m_o^2c^4$$

$$E^2 = P^2c^2 + E_o^2$$

Variation of mass with velocity

Consider a particle of rest mass m_o and charge q moving with velocity v in electron



$$F_{mg} = qvB$$

q is the electric charge , v the velocity of the electron , B is the magnetic force

$F_{cent} = \frac{mv^2}{r}$ Where r is the radius of the circle ,and m is the moving mass

$$qvB = \frac{mv^2}{r}$$

$$\frac{q}{m_o} = \frac{\gamma v}{rB}$$

Example

An electron is accelerated from rest through a potential difference

$\Delta V = 10^7$ volts find

a) Its kinetic energy b) its total energy c) it's mass d) its speed e) its momentum.

Solution

a)K=k_f-k_i=k_f-0=$e\Delta V = ex10^7 V = 10eV = 10MeV$

b)E=k+m_oc^2=10 MeV+0.5 MeV=10.5 MeV

$$k = (\gamma - 1)m_o c^2$$

$$\gamma = \frac{k}{m_o c^2} + 1 = \frac{10}{0.5} + 1 = 21$$

c)Since the rest mass of the electron m_o=9.1x10^{-31} kg

$$m = \gamma m_o = 21x9.1x10^{-31} kg = 1.91x10^{-29} kg$$

d)

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 21$$

$$\therefore v = 0.9983c$$

e) P=mv=$\gamma m_o v$

=21x9.1 x 10^{-31} kgx 0.9983x 3x10^8 m/s

Or we could say

$$P = \sqrt{\frac{E^2 - E_o^2}{c^2}} = \sqrt{\frac{(10.5)^2 - (0.5)^2}{c^2}} MeV = 10.485 \frac{MeV}{c}$$

Or assuming an extreme relativistic case, we would have

$$P \approx \frac{E}{c} = 10.5 \frac{MeV}{c}$$

Example

According to an observer on earth, a car covers 1 mile on 50 s on a straight stretch of high way. How long will it take to cover this distance according to an observer with his own clock on a spaceship which is receding from earth with a speed of 0.95 c a) perpendicular to the line of motion of the car, and b) along the same line of the motion as the car?

Example

Consider two rockets, A, and B, each moving with velocity 0.9 c relative to the earth, and approaching each other according to an observer standing on the earth.

Calculate the relative velocity of B with respect to A.

Example

Find the mass of an electron $m_o = 9.1 \times 10^{-31}$ kg whose velocity is 0.99 c.

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Homework

Q1- Spacecraft Alpha has a velocity with respect to the earth of 0.9 C, if spacecraft Beta is to pass Alpha at a relative velocity of 0.5 C. What velocity must Beta have with respect to the earth?

Q2- A rod 1m length is lying along the x-axis in the inertial system S. this length is being measured by an observer in S' which is moving along xx' axis with a velocity v. What should the value of v be so that to an observer in S' the length of the rod looks to be o.5 m?

Q3-The time period T_o of a pendulum is 2 s, it takes 2 s to complete one cycle. What is the time period of this pendulum when measured by an observer whose inertial system is moving with a speed of 0.9 C with respect to the inertial system of the pendulum?

Q4- Calculate the momentum and the total energy of a proton moving with a speed of 0.9C.

Q5- A particle has a kinetic energy equal to half its rest mass energy. Calculate the speed and momentum of a particle.

Q6-Calculate the mass and the momentum of a 1-GeV electron.