Homogenous coordinate in Transformation Matrix

Why Homogeneous Coordinates?

1. Mathematicians commonly use homogeneous coordinates as they allow scaling factors to be removed from equations.

2. Using homogeneous coordinates allows us use matrix multiplication to calculate transformations – extremely efficient!

3. Homogeneous coordinates seem unintuitive, but they make graphics operations much easier

4. Since a 2x2 matrix representation of translation does not exist!!. So by using a homogenous coordinate system then we can represent *2x2 translation transformation* as a matrix multiplication.

5. It provides a consistent, uniform way of handling *affine transformations*. 2D affine transformations always have a bottom row of **[0 0 1].**

An "affine point" is a "linear point" with an added w-coordinate which is always 1:

$$p_{aff} = \begin{bmatrix} p_{lin} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Applying an affine transformation gives another affine point.

A point (x, y) can be re-written in homogeneous coordinates as (xw, yw,w)

- The **homogeneous parameter** w is a non- zero value such that x and y coordinates can easily be recovered by dividing the first and second numbers by the third.

$x = \frac{xw}{w}$ $y = \frac{yw}{w}$

- We can then write any point (x, y) as :

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} w.x \\ w.y \\ w \end{bmatrix}, w \neq 0$$

- We can conveniently choose w = 1 so that (x, y) becomes:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

In homogeneous coordinates the **scaling matrix** as follows:

$$\begin{bmatrix} sx & 0 \\ 0 & sy \end{bmatrix} \rightarrow \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} xw \\ yw \\ w \end{bmatrix} = \begin{bmatrix} sxxw \\ syyw \\ w \end{bmatrix} \rightarrow \text{divide by w then}$$

$$\begin{bmatrix} s_x x \\ s_y y \end{bmatrix} \textit{is the correctly scaled point}$$

The counterclockwise **rotation matrix** is:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Using homogeneous coordinates we get:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying it to the point (x,y) with homogeneous (xw,yw,w) gives:

$$R.P = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} xw \\ yw \\ w \end{bmatrix}$$

The homogeneous coordinate **translation matrix** of t_x , t_y is

$$T = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix}$$

$$T.P = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} xw \\ yw \\ w \end{bmatrix} = \begin{bmatrix} (xw + txw) \\ (yw + tyw) \\ w \end{bmatrix} \rightarrow$$

divide by w then we get the point p=$\begin{bmatrix} x + tx \\ y + ty \end{bmatrix}$ is the correctly translated point.

In general

1. Translation

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2. Rotation

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3. Scaling

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ 1 \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation about an arbitrary point

- The homogeneous coordinate transformation matrix for counterclockwise rotation about point (xc , yc) is done by three steps as follows:

1- Translation to the origin

$$T1 = \begin{bmatrix} 1 & 0 & -xc \\ 0 & 1 & -yc \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -xc \\ 0 & 1 & -yc \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2- Rotation about the origin

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \bar{\bar{x}} \\ \bar{\bar{y}} \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ 1 \end{bmatrix}$$

3- Translation back to its correct position

$$T2 = \begin{bmatrix} 1 & 0 & xc \\ 0 & 1 & yc \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \equiv x \\ \equiv y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & xc \\ 0 & 1 & yc \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\bar{x}} \\ \bar{\bar{y}} \\ 1 \end{bmatrix}$$

H.W Rotate the triangle (5,2),(1,1),(0,0) by rotation in counterclockwise with angle 45° about fixed point (-1,-1)?

Composite Transformations

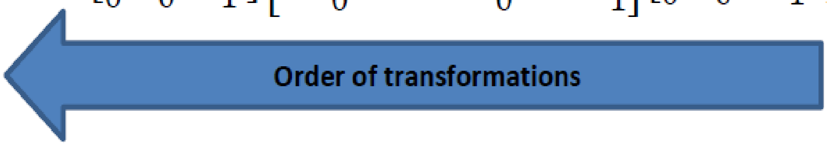
- Matrices are a convenient and efficient way to represent a sequence of transformations.
- Matrix multiplication is not commutative so that the order of transformations is important.
- What if we want to rotate and translate?

To rotate a point $(T_2\,(R\,(T_1 \begin{bmatrix} xw \\ yw \\ w \end{bmatrix})))$

- Now we must form an overall transformation matrix as follows:-

$$(T_2(R\,T_1)) \begin{bmatrix} xw \\ yw \\ w \end{bmatrix}$$

$$T_2 R\, T_1 = \begin{bmatrix} 1 & 0 & x_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Cos(\theta) & -Sin(\theta) & 0 \\ Sin(\theta) & Cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & 1 \end{bmatrix}$$



$$= \begin{bmatrix} 1 & 0 & x_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Cos(\theta) & -Sin(\theta) & -x_c\,Cos(\theta) + y_c Sin(\theta) \\ Sin(\theta) & Cos(\theta) & -x_c\,\sin(\theta) - y_c Cos(\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} Cos(\theta) & -Sin(\theta) & -x_c Cos(\theta) + y_c Sin(\theta) + x_c \\ Sin(\theta) & Cos(\theta) & -x_c Sin(\theta) - y_c Cos(\theta) + y_c \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} Cos(\theta) & -Sin(\theta) & x_c(1 - Cos(\theta)) + y_c Sin(\theta) \\ Sin(\theta) & Cos(\theta) & y_c(1 - Cos(\theta)) - x_c Sin(\theta) \\ 0 & 0 & 1 \end{bmatrix}$$