معادلة كوبلاس

1- using Laplace transformation to solve the Initial value problem.

Example:

By using Laplace transform solve the following D.Es.

① $y' - y = e^{-t}$, $y(0) = 1$

Solution:

$$\mathcal{L}\{y' - y\} = \mathcal{L}\{e^{-t}\}$$

$$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \frac{1}{s+1}$$

$$s\mathcal{L}\{y\} - y(0) - \mathcal{L}\{y\} = \frac{1}{s+1}$$

$$s\mathcal{L}\{y\} - 1 - \mathcal{L}\{y\} = \frac{1}{s+1}$$

$$\mathcal{L}\{y\}(s-1) = \frac{1}{s+1} + 1$$

$$\mathcal{L}\{y\}(s-1) = \frac{1+s+1}{s+1}$$

$$\mathcal{L}\{y\} = \frac{s+2}{(s+1)(s-1)}$$

$$y = \mathcal{L}^{-1}\left\{ \frac{s+2}{(s^2-1)} \right\}$$

$$\therefore y = \mathcal{L}^{-1}\left\{ \frac{s}{s^2-1} \right\} + \mathcal{L}^{-1}\left\{ \frac{2}{s^2-1} \right\}$$

$$\therefore y = \cosh t + 2 \sinh t .$$

Example (2)

$$y'' - 3y' + 4y = 0$$

$y(0) = 1$
$y'(0) = 5$

Solution:

$$\mathcal{L}\{ y'' - 3y' + 4y \} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 0$$

$$s^2\mathcal{L}\{y\} - s y(0) - y'(0) - 3[s\mathcal{L}\{y\} - y(0) + 4\mathcal{L}\{y\} = 0$$

$$s^2\mathcal{L}\{y\} - s(1) - 5 - 3s\mathcal{L}\{y\} + 3 + 4\mathcal{L}\{y\} = 0$$

$$s^2\mathcal{L}\{y\} - 3s\mathcal{L}\{y\} + 4\mathcal{L}\{y\} = s + 2$$

$$\mathcal{L}\{y\}(s^2 - 3s + 4) = s + 2$$

$$\mathcal{L}\{y\} = \frac{s+2}{s^2-3s+4}$$

$$y = \mathcal{L}^{-1}\left\{ \frac{s+2}{s^2-3s+4} \right\}$$

$$y = \mathcal{L}^{-1}\left\{ \frac{s+2}{s^2-3s+\frac{9}{4}+\frac{7}{4}} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{s+2}{(s-\frac{3}{2})^2+\frac{7}{4}} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{s-\frac{3}{2}+\frac{3}{2}}{(s-\frac{3}{2})^2+\frac{7}{4}} \right\} + \mathcal{L}^{-1}\left\{ \frac{2}{(s-\frac{3}{2})^2+\frac{7}{4}} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{s-\frac{3}{2}}{(s-\frac{3}{2})^2+\frac{7}{4}} \right\} + \mathcal{L}^{-1}\left\{ \frac{\frac{3}{2}}{(s-\frac{3}{2})^2+\frac{7}{4}} \right\} +$$

$$\mathcal{L}^{-1}\left\{ \frac{2}{(s-\frac{3}{2})^2+\frac{7}{4}} \right\}$$

$$= \frac{2}{\sqrt{7}} e^{3/2} \cos \frac{\sqrt{7}}{2} x + \frac{3}{\sqrt{7}} e^{\frac{3}{2}} e^{\frac{3}{2}} \sin \frac{\sqrt{7}}{2}$$

$$+ \frac{4}{\sqrt{7}} \sin \frac{\sqrt{7}}{2}.$$

③ $y'' - 2y' = 6 - 4t$

Solution:

$y(0) = 2$

$y'(0) = 0$

$$\mathcal{L}\{y'' - 2y'\} = \mathcal{L}(6 - 4t)$$

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} = \mathcal{L}(6) - \mathcal{L}(4t)$$

$$s^2\mathcal{L}(y) - sy(0) - y'(0) - 2[s\mathcal{L}(y) - y(0)] = \frac{6}{s} - \frac{4}{s^2}$$

$$s^2\mathcal{L}(y) - 2s - 2s\mathcal{L}\{y\} + 4 = \frac{6}{s} - \frac{4}{s^2}$$

$$s^2\mathcal{L}\{y\} - 2s\mathcal{L}\{y\} = \frac{6s-4}{s^2} + 2s - 4$$

$$\mathcal{L}\{y\}(s^2 - 2s) = \frac{6s - 4 + 2s^3 - 4s^2}{s^2}$$

$$y = \mathcal{L}^{-1}\left\{\frac{6s - 4 + 2s^3 - 4s^2}{s^2(s^2 - 2s)}\right\}$$

$\downarrow$

H.W

(2) using Laplace transform to solve systems of D.Es:

$$x' + 4y = 0 \quad \text{---} (*)$$
$$x + y' = t \quad \text{---} (**)$$

$$\left\{ \begin{array}{l} y(0) = 1 \\ x(0) = -1 \end{array} \right.$$

Solution:

To solve (*)

$$\mathcal{L}\{x' + 4y\} = \mathcal{L}\{0\}$$

$$\Rightarrow \mathcal{L}\{x'\} + \mathcal{L}\{4y\} = 0$$

$$s\mathcal{L}\{x\} - x(0) + 4\mathcal{L}\{y\} = 0$$

$$s\mathcal{L}\{x\} + 1 + 4\mathcal{L}\{y\} = 0$$

$$s\mathcal{L}\{x\} + 4\mathcal{L}\{y\} = -1 \quad \text{---} (1)$$

To solve **

$$\mathcal{L}\{x + y'\} = \mathcal{L}\{t\}$$

$$\mathcal{L}\{x\} + \mathcal{L}\{y'\} = \mathcal{L}\{t\}$$

$$\Rightarrow \mathcal{L}\{x\} + s\mathcal{L}\{y\} - y(0) = \frac{1}{s^2}$$

$$\mathcal{L}\{x\} + s\mathcal{L}\{y\} - 1 = \frac{1}{s^2}$$

$$\Rightarrow \mathcal{L}\{x\} + s\mathcal{L}\{y\} = \frac{1}{s^2} + 1$$

$$\mathcal{L}\{x\} + s\mathcal{L}\{y\} = \frac{1+s^2}{s^2} \quad \text{---} \; (2)$$

$$s\mathcal{L}\{x\} + 4\mathcal{L}\{y\} = -1$$ نضرب بالإشارات

$$\mathcal{L}\{x\} + s\mathcal{L}\{y\} = \frac{1+s^2}{s^2}$$ نضرب (s)

$$\mp s\mathcal{L}\{x\} \mp 4\mathcal{L}\{y\} = \pm 1$$

$$s\mathcal{L}\{x\} + s^2\mathcal{L}\{y\} = \frac{1+s^2}{s}$$

$$s^2\mathcal{L}\{y\} - 4\mathcal{L}\{y\} = 1 + \frac{1+s^2}{s}$$

$$\mathcal{L}\{y\}(s^2-4) = \frac{s+1+s^2}{s}$$

$$\mathcal{L}\{y\} = \frac{s+1+s^2}{s(s^2-4)}$$

$$y = \mathcal{L}^{-1}\left\{ \frac{s+1+s^2}{s(s^2-4)} \right\}$$ تجزئة بالكسور

$$y = \mathcal{L}^{-1}\left\{\frac{A}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{B}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{c}{s+2}\right\}$$

$$A = \frac{-1}{4} \quad , \quad B = \frac{7}{8} \quad , \quad c = \frac{3}{8}$$

$$= \mathcal{L}^{-1}\left\{\frac{-1/4}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{7/8}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{3/8}{s+2}\right\}$$

$$\therefore y = \frac{-1}{4} + \frac{7}{8} e^{2t} + \frac{3}{8} e^{-2t}$$

$$x = t - 2\left(\frac{7}{8}\right) e^{2t} + 2\left(\frac{3}{8}\right) e^{-2t}$$

$$x = t - \frac{7}{4} e^{2t} + \frac{3}{4} e^{-2t} .$$

① قيم كل من x و y بدلالة الـ t

② نجد قيمة x من خلال المعادلة التي تحتوي على x .

② $x'' + y' = \cos t$ --- * $\quad x(0) = -1 \quad , \quad x'(0) = -1$

$y'' - x = \sin t$ --- ** $\quad y(0) = 1 \quad , \quad y'(0) = 0$

solution:

To solve *

$$\mathcal{L}\{x''\} + \mathcal{L}\{y'\} = \mathcal{L}\{\cos t\}$$

$$s^2\mathcal{L}\{x\} - sx(0) - x'(0) + s\mathcal{L}\{y\} - y(0) = \frac{s}{s^2+1}$$

$$s^2\mathcal{L}\{x\} + s + \cancel{1} + s\mathcal{L}\{y\} - \cancel{1} = \frac{s}{s^2+1}$$

$$s^2\mathcal{L}\{x\} + s\mathcal{L}\{y\} = \frac{s}{s^2+1} - s$$

$$\therefore s^2\mathcal{L}\{x\} + s\mathcal{L}\{y\} = \frac{\cancel{s} - s^3 \cancel{- s}}{s^2+1}$$

$$\Rightarrow s^2\mathcal{L}\{x\} + s\mathcal{L}\{y\} = \frac{-s^3}{s^2+1} \quad \text{--- } ①$$

To solve **

$$\mathcal{L}\{y''\} - \mathcal{L}\{x\} = \mathcal{L}\{\sin t\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - \mathcal{L}\{x\} = \frac{1}{s^2+1}$$

$$s^2\mathcal{L}\{y\} - s - \mathcal{L}\{x\} = \frac{1}{s^2+1}$$

$$s^2\mathcal{L}\{y\} - \mathcal{L}\{x\} = \frac{1}{s^2+1} + s$$

$$\Rightarrow s^2\mathcal{L}\{y\} - \mathcal{L}\{x\} = \frac{1+s^3+s}{s^2+1} \quad --- (2)$$

$$\therefore \quad s^2\mathcal{L}\{x\} + s\mathcal{L}\{y\} = \frac{-s^3}{s^2+1}$$

$$s^2\mathcal{L}\{y\} - \mathcal{L}\{x\} = \frac{1+s^3+s}{s^2+1} \qquad * s^2$$

Hence

$$\cancel{s^2\mathcal{L}\{x\}} + s\mathcal{L}\{y\} = \frac{-s^3}{s^2+1}$$

$$s^4\mathcal{L}\{y\} - \cancel{s^2\mathcal{L}\{x\}} = \frac{(1+s^3+s)s^2}{s^2+1}$$

$$s^4\mathcal{L}\{y\} + s\mathcal{L}\{y\} = \frac{-s^3}{s^2+1} + \frac{(1+s^3+s)s^2}{s^2+1}$$

$$\mathcal{L}\{y\}(s^4+s) = \frac{\cancel{-s^3}+s^2+s^5\cancel{+s^3}}{s^2+1}$$

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$$\mathcal{L}\{y\} = \frac{s^5 + s^2}{(s^2+1)(s^4+s)}$$

$$\therefore y = \mathcal{L}^{-1}\left\{ \frac{s^5+s^2}{(s^2+1)(s^4+s)} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{s\cancel{(s^4+s)}}{(s^2+1)\cancel{(s^4+s)}} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{s}{(s^2+1} \right]$$

$$\therefore y = \cos t$$

$$\therefore x = y'' - \sin t$$

$$y' = -\sin t$$

$$y'' = -\cos t$$

Hence

$$x = -\cos t - \sin t .$$

(3) $y'' + x + y = 0 \qquad y(0)=0,\ y'(0)=0$

$x' + y' = 0 \qquad x(0)=1$

$\mathcal{L}\{y''\} + \mathcal{L}\{x\} + \{y\} = 0$

$s^2\mathcal{L}(y) - s\,y(0) - y'(0) + \mathcal{L}\{x\} + \mathcal{L}\{y\} = 0$

$\mathcal{L}\{y\}(s^2+1) + \mathcal{L}\{x\} = 0 \quad ---(1)$

To solve $**$

$\mathcal{L}\{x'\} + \mathcal{L}\{y'\} = 0$

$s\mathcal{L}\{x\} - x(0) + s\mathcal{L}\{y\} - y(0) = 0$

$\Rightarrow s\mathcal{L}\{x\} - 1 + s\mathcal{L}\{y\} = 0$

$\Rightarrow s\mathcal{L}\{x\} + s\mathcal{L}\{y\} = 1 \quad --(2)$

Hence

$\mathcal{L}\{y\}(s^2+1) + \mathcal{L}\{x\} = 0 \qquad * s$

$\mp s\cancel{\mathcal{L}\{x\}} \mp s\mathcal{L}\{y\} = \mp 1 \quad ——(2)$

$\mathcal{L}\{y\}\,s(s^2+1) + s\cancel{\mathcal{L}\{x\}} = 0 \quad ——(1)$

$$\mathcal{L}\{y\}\, s(s^2+1) - s\mathcal{L}\{y\} = -1$$

$$\mathcal{L}\{y\}(s^3 + \cancel{s} - \cancel{s}) = -1$$

$$\mathcal{L}\{y\}\, s^3 = -1$$

$$\mathcal{L}\{y\} = \frac{-1}{s^3} \quad \Rightarrow y = \mathcal{L}^{-1}\left\{\frac{-1}{s^3}\right\}$$

$$\Rightarrow y = \frac{-1}{2} t^2$$

$$\therefore x = -y - y''$$

$$\Rightarrow = \frac{1}{2}t^2 + 1$$

Problem

By using Laplace transform solve the following :-

1. $y' + y = e^{2t} \quad , y(0) = 0$

2. $y'' + 2y' + y = 3t e^{-t} \quad , y(0) = 4 , y'(0) = 2$

3. $y'' - 3y' + 2y = e^{3t} \quad y(0) = 0 , y'(0) = 0$

4. $y'' + y = 6 \sin 2t \quad y(0) = -1 , \; y'(0) = -4$

5. $y'' + 3y' + 2y = 4t^2 \quad , y(0) = 0 , y'(0) = 0$

6. $y'' - 4y' + 4y = 4 \cos 2t \quad , y(0) = 2 , \; y'(0) = 5$

7. $y'' - 2x = 2$
   $y + x' = 5e^{2t} + 1 \quad y(0) = 2 , \; x(0) = 1 , y'(0) = 2$

8. $y' + x'' = e^{t}$
   $x - y = 2 \quad y(0) = 0 , \; x(0) = 0 , x'(0) = 1$

بعض المتطابقات والقوانين

1- $\sin 2x = 2\sin x \cos x$

2- $\sin 4x = 2\sin 2x \cos 2x$

3- $\sin^2 \alpha x = \frac{1}{2}(1-\cos 2\alpha x)$

4- $\cos^2 \alpha x = \frac{1}{2}(1+\cos 2\alpha x)$

5- $\cos 4x = \cos^2 2x - \sin^2 2x$

6- $\cos 3x = \cos^2 \frac{3}{2}x - \sin^2 \frac{3}{2}x$

7- $\sin x \cos x = \frac{1}{2}\sin 2x$

8- $\sinh \alpha x = \frac{e^{\alpha x} - e^{-\alpha x}}{2}$

9- $\cosh \alpha x = \frac{e^{\alpha x} + e^{-\alpha x}}{2}$

examples

① $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-5)^3}\right\}$

$$\because \mathcal{L}\{t^2\} = \frac{2}{s^3} \quad \text{and} \quad \mathcal{L}\{e^{5t} \cdot x^2\} = \frac{2}{(s-5)^3}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s-5)^3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-5)^3}\right\} = \frac{1}{2}e^{5t}t^2$$

② $\mathcal{L}^{-1}\left\{\dfrac{2(s+1)}{(s+1)^2+9}\right\}$

$$\because \mathcal{L}\{\cos 3t\} = \frac{s}{s^2+9} \quad \text{and} \quad \mathcal{L}\{e^{-t}\cos 3t\} = \frac{(s+1)}{(s+1)^2+9}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{2(s+1)}{(s+1)^2+9}\right\} = 2\mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+9}\right\} = 2e^{-t}\cos 3t.$$

(3) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+29}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+4+25}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+25}\right\}$

$= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^2+25}\right\}$

$= \frac{1}{5} e^{-2t} \sin 5t.$

(4) $\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{2s+3}{s^2-2s+1+4}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{2s-2+5}{(s\cancel{+1})^2+4}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{2(s-1)+5}{(s-1)^2+4}\right\}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-1)^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{5}{(s-1)^2+4}\right\}$

$= 2e^{t}\cos 2t + \frac{5}{2}e^{t}\sin 2t.$

(5) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s+3)(s+5)}\right\}$

$$\frac{s}{(s+3)(s+5)} = \frac{A}{(s+3)} + \frac{B}{(s+5)}$$

$$= \frac{A(s+5) + B(s+3)}{(s+3)(s+5)}$$

Now $As + A5 + Bs + 3B = s$

and $\left.\begin{array}{l} A + B = 1 \\ 5A + 3B = 0 \end{array}\right\} \begin{array}{l} * \, 3 \\ \cancel{*} \end{array}$

$$\begin{array}{l} 3A + \cancel{3B} = 3 \\ \mp 5A \mp \cancel{3B} = \mp 0 \\ \hline -2A = 3 \Rightarrow A = -\frac{3}{2} \text{ and } B = \frac{5}{2} \end{array}$$

Then

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+3)(s+5)}\right\} = \mathcal{L}^{-1}\left\{\frac{-3/2}{(s+3)}\right\} + \mathcal{L}^{-1}\left\{\frac{5/2}{(s+5)}\right\}$$

$$= -\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+3)}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+5)}\right\}$$

$$= -\frac{3}{2}e^{-3t} + \frac{5}{2}e^{-5t}.$$

⑥ $\mathcal{L}^{-1}\left\{\dfrac{2s^2}{(s-1)(s^2+1)}\right\}$

$$\frac{2s^2}{(s-1)(s^2+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{s^2+1}$$

$$= \frac{A(s^2+1) + (Bs+C)(s-1)}{(s-1)(s^2+1)}$$

Now

$$As^2 + A + Bs^2 - Bs + Cs - C = 2s^2$$

$\therefore \quad A+B=2 \; , \; C-B=0 \; , \; A-C=0$

$\therefore \left.\begin{array}{l} A+C=2 \\ A-C=0 \end{array}\right\} \Rightarrow 2A=2 \Rightarrow A=1, \; B=1 \; , C=1$

$$\therefore \mathcal{L}^{-1}\left\{\frac{2s}{(s-1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)}\right\} + \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^t + \cos t + \sin t .$$