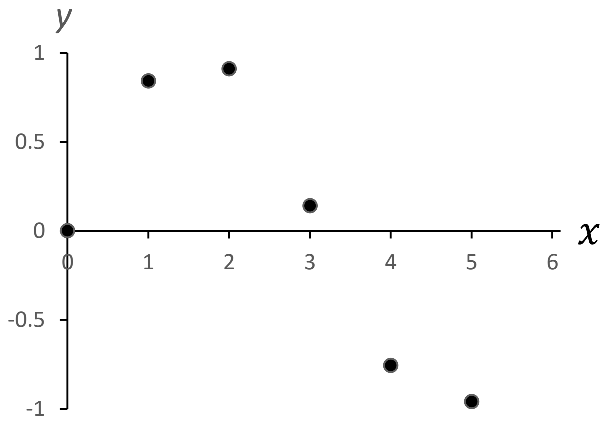
Interpolation, Extrapolation, and Inverse Interpolation

Interpolation is a method of constructing new data points within the range of discrete set of known data points.

In engineering and science one often has a number of data points, are obtained by sampling or experimentation, and try to construct a function which closely fits those data points. This is called "Curve Fitting" or "Regression" analysis. Interpolation is a specific case of curve fitting in which the function must go exactly through the data points.

Example: Suppose we have a table which gives some values of an unknown function f.

x	$f(x)$
0	0
1	0.8415
2	0.9093
3	0.1411
4	-0.7568
5	-0.9589
6	-0.2794



Note: There are many different interpolation methods.

Question

- How accurate is the method?
- How expensive is it?
- How smooth is the interpolation?
- How many data point are needed?

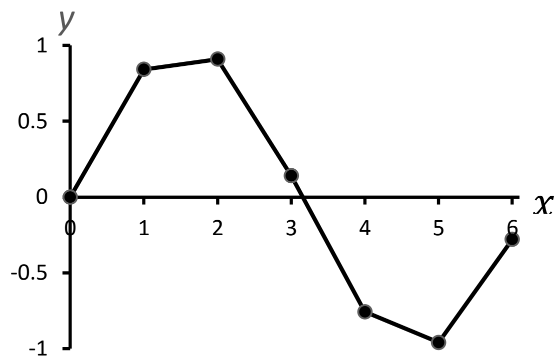
1. Linear interpolation

One of the simplest methods is linear interpolation. Consider the above example of determining $f(2.5)$. since 2.5 is midway between 2 and 3. It is reasonable to take $f(2.5)$ midway between $f(2) = 0.9093$ and $f(3) = 0.1411$, which yields 0.5252.

Generally, linear interpolation takes two data points, say (x_a, y_a) and (x_b, y_b), and the interpolante is given by:

$$y = y_0 + (x - x_0)\frac{(y_b - y_a)}{(x_b - x_a)} \quad \text{at the point } (x, y)$$

Linear interpolation is:



- quick, and
- Easy, but
- It is not very precise.
- Another disadvanage is that the interpolation is not differenable at the point x_k.

In words, the error is proportional to the square of the distance between the data points.

2. Polynomial interpolation

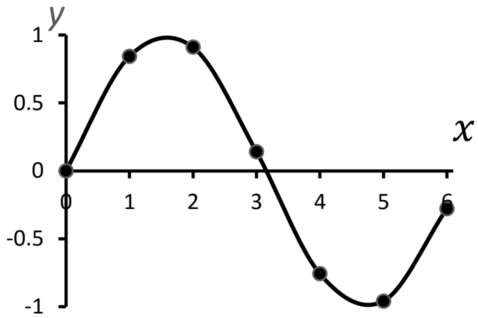
Polynomial interpolation is a generalization of linear interpolation. Note that the linear interpolant is a linear function, we now replace this interpolants by a polynomial of higher degree.

The following six degree polynomial goas through all the seven points:

$$f(x) = -0.00012521x^6 - 0.003130x^5 + 0.07321x^4 - 0.3577x^3 + 0.2255x^2 + 0.9038x$$

Substituting $x = 2.5$, we find that

$$f(2.5) = 0.5965$$



Generally, if we have n data points, there is exactly one polynomial of degree at most $n-1$ going through all the data points.

The interpolant Error is proportional to the distance between the data points to the power n.

Furthermore, the interpolant is a polynomial and thus infinitely differentiable. We see that polynomial interpolation solves all the problems of linear interpolation.

However, polynomial interpolation also has some disadvantages:

⇒ May not be exact after all, especially at the end points.

⇒ These disadvantages can be avoided by using spline interpolation.

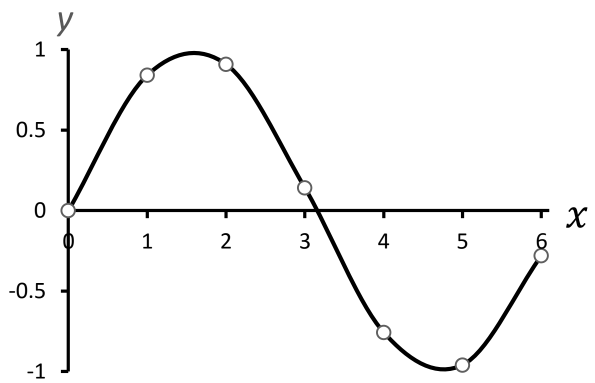
3. Spline Interpolation

Remember that linear interpolation uses a linear function for each of the intervals $[x_k, x_{k+1}]$. Spline interpolation uses low degree polynomials in each of the intervals, and chooses the polynomial pieces such that they fit smoothly together. The resulting function is called a "Spline"

The natural cubic spline is:

- Piecewise cubic, and
- Twice continuously differentiable.
- It's 2nd derivative is zero at the end points.

x	$f(x)$
0	0
1	0.8415
2	0.9093
3	0.1411
4	-0.7568
5	-0.9589
6	-0.2794



$$f(x) = \begin{cases} -0.1522x^3 + 0.9937x & if\ x \in [0,1] \\ -0.01258x^3 - 0.4189x^2 + 1.4126x - 0.1396 & if\ x \in [1,2] \\ 0.1403x^3 - 1.3359x^2 + 3.2467x - 01.3623 & if\ x \in [2,3] \\ 0.1579x^3 - 1.4945x^2 + 3.7225x - 1.8881 & if\ x \in [3,4] \\ 0.05375x^3 - 0.2450x^2 - 1.2756x + 4.8259 & if\ x \in [4,5] \\ -0.181x^3 + 3.3673x^2 - 19.3370x - 34.9282 & if\ x \in [5,6] \end{cases}$$

In this case we get:

$$f(2.5) = 0.5972$$

Like "Polynomial interpolation", "spline interpolation" incurs a smaller error that linear interpolation and the interpolant is smooth. However, the interpolant is easier to evaluate than the high-degree polynomials used in polynomial interpolation.

Application

"Audio Signal Processing terms"

Extrapolation

Extrapolation is used if we want to find data points outside the range of known data points.

External links

1. Dotplacer applet

    * (Applet showing various interpolation methods, with movable points)

    Website

2. http://numericalmethodes.eng.usf.edu

Interpolating Polynomial

Question: why do we use polynomials?

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

Answer: the algebraic polynomials $P_n(x)$ are by far the most important and popular approximation for the following reasons:

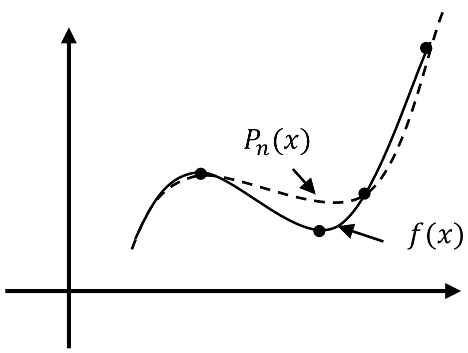
1. The theory of polynomial approximation is well-developed and fairly simple.
2. Polynomials can be easy to evaluate and their sums, products and differences are also polynomials.
3. Polynomials can be differentiated and integrated with little difficulty, yield other polynomial in both cases.
4. If the origin of the coordinate system is shifted or if the scale of the independent variable is changed, the transformed polynomials remain polynomials. That is, if $P_n(x)$ is a polynomial so are $P_n(x + a)$ and $P_n(xa)$.

⟶ given a pair of values $[x_i, f(x_i)]$ or $[x_i, y_i]$ $i = 0,1,2, \dots, n$.

In order to determine the coefficients of $P_n(x)$, it is required to consider that;

$$\boxed{P_n(x_i) = f(x_i)} \quad i = 0,1,2, \dots, n$$

Thus the nth-degree polynomial $P(x)$ must reproduce $f(x)$ exactly for the $(n + 1)$ arguments $x = x_i$.



Note: the required that:

$P(x_i) = f(x_i), \quad i = 0,1,2, \dots, n$

Established the value of $P_n(x)$ for all x, but in no way guarantee accurate approximation of $f(x)$ for $x \neq x_i$, that is, for arguments other than the given base points. If $f(x)$ should be a polynomial of degree (n) or less, agreement is ofcource exact for all x.

Question: Define interpolation, give the general proposal of it?

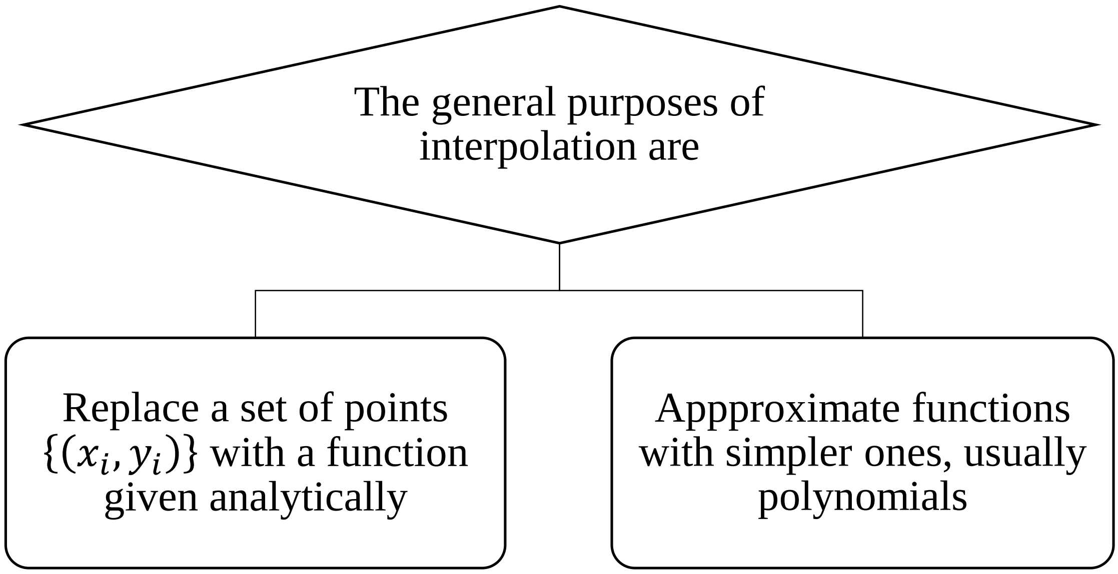
Answer:

1. Interpolation is a process of finding a formula (often a polynomial) whose graph will pass through a given set of points (x, y).
2. Interpolation is a rational process, generally used in estimating a missing functional value by making a weight average of known functional valued at neighboring points.

Question: In dealing with tabular quantities, three basic problems are encountered??

Answer:

1. Given a mathematical relationship in tabular form, one may wish to extend its range beyond that given by the original data.
2. One may wish to approximate a functional value between two data points.
3. One may wish to approximate an independent variable corresponding to a given functional value.



Purpose #1: has several aspects:

- The data may be form a known class of functions, interpolation is the used to find the member of this class of functions that agree with the given data.

  For Example, data may be generated from functions of the form:

$$P(x) = a_0 + a_1 e^x + a_2 e^{2x} + \cdots + a_n e^{nx}$$

Then we need to find the coefficients $\{a_{ij}\}$ based on the given data values.

We may want to take function values $f(x)$ given in a table for selected values of x, often equally spaced, and extend the function to values of x not in the table.

For example, given number from a table of logarithms, estimate the logarithm of the number x not in the table.

- Given a set of data $\{(x_i, y_i)\}$, find a curve passing through these points, that is , "pleasing to the eye".

  In fact, this is what is done continuously with computer graphics.

Question: How do we connect a set of points to make a smooth curve?

Connecting them with straight line segments will often give a curve with many corners, whereas what was intended was a smooth curve.

Purpose #2:

For interpolation is to approximate function $f(x)$ by simpler functions $P(x)$ perhaps to make it easier to integrate or differentiate $f(x)$.

$$I = \int_0^1 \frac{dx}{1 + x^{10}}$$

This is very difficult to do *analytically*. But we will look at producing polynomial interpolants of the integrand; and polynomials are easily integrated *Exactly*.

Classification of Interpolating Polynomial

Can be classified into one of two groups:

1. Those applicable for arbitrary spaced base points $(x_0, x_1, \dots)$.
2. Those for evenly spaced points, that is for base points:

$$x_0, x_1 = x_0 + h, x_2 = x_0 + 2h$$

$$\boxed{x_n = x_0 + nh}$$

Where, h is the constant spacing or step size between adjacent ordered x_i values.

The two most common forms of the interpolating polynomial for arbitrary spaced based points are:

1. Newton's Divided difference interpolating polynomials.
2. Lagrang's interpolating polynomial.

Both permit arbitrary ordering of the base points $x_0, x_1, x_2, \dots, x_n$ as well.

Interpolation methods ⇒

- Direct interpolation
- Newton Divided Difference
- Lagrange's interpolation
- Spline interpolation

∗ Finite-Differences (Newton Divided difference) may be used in:

⇒ Interpolation,

⇒ Curve Fitting,

⇒ Differentiation, and

⇒ Integration.

Note: Duplicate base-point values are not permitted since it well lead to no unique solution.

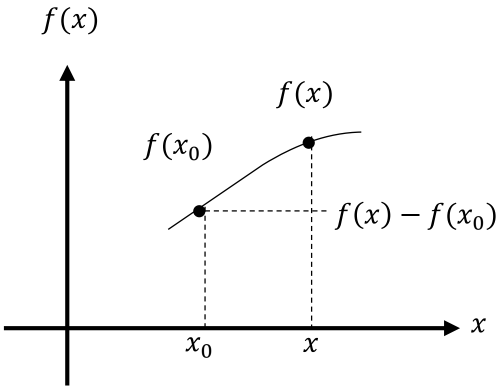
The square-interval formulas had been developed to simplify interpolation in tables of functions with evenly spaced arguments. Some has been developed specifically for interpolation near the beginning, middle, or end of a table.

All can be derived from either Lagrange's or Newton's divided differences interpolating polynomial.

Newton's Divided-difference interpolating

Definition of derivatives:

$$\left[\frac{df}{dx}\right]_{x=x_0} = f'(x_0) = \lim_{x\to 0} \frac{f(x)-f(x_0)}{x-x_0}$$



For finite or discrete mathematics, it is useful to define as approximation to the derivative,

$$\boxed{f[x,x_0] = \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)} \text{ provided } x \neq x_0$$

$f[x,x_0] \Longrightarrow$ 1st finite divided difference, or just finite difference of order 1 relative to arguments x, x_0.

Note: First finite divided difference and the first derivative is clearly indicated by the differential mean value theory from elementary calculus. **How!**

Table1

The finite divided-difference table

Order	Difference Notation	Definition
0	$f[x_0]$	$f[x_0]$
1	$f[x_1,x_0]$	$\frac{f(x_1)-f(x_0)}{x_1-x_0}$
2	$f[x_2,x_1,x_0]$	$\frac{f[x_2,x_1]-f[x_1,x_0]}{x_2-x_0}$
3	$f[x_3,x_2,x_1,x_0]$	$\frac{f[x_3,x_2,x_1]-f[x_2,x_1,x_0]}{x_3-x_0}$
⋮	⋮	⋮ ⋮

n	$f[x_n, x_{n-1}, x_{n-2}, \dots, x_0]$	$\frac{f[x_n, x_{n-1}, x_{n-2}, \dots, x_1] - f[x_{n-1}, x_{n-2}, \dots, x_0]}{x_n - x_0}$

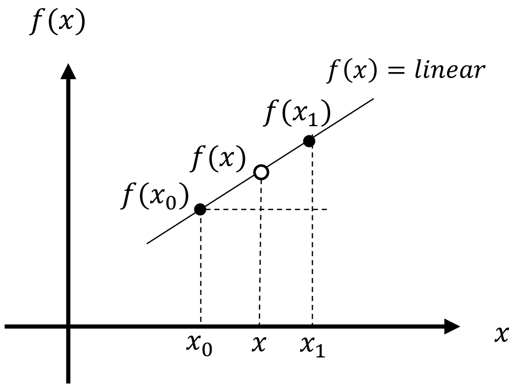
Linear interpolation

The simplest form of interpolation is probably the straight line, connecting two points by a straight line. Let two data points, (x_0, y_0) and(x_1, y_1) be given. There is a unique straight line passing through these points.

What is the polynomial function $f(x) \equiv P_1(x)$

Given only the base point information

0	x_i	$f(x_i)$
1	x_1	$f(x_1)$
2	x_2	$f(x_2)$



$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x - x_0}$$

From geometrical consideration alone, it is apparent that for this case:

$$f[x, x_0] = f[x_1, x_0]$$

$$f[x, x_0] = \frac{f[x] - f[x_0]}{x - x_0} = f[x_1, x_0]$$

$$\frac{f[x] - f[x_0]}{x - x_0} = f[x_1, x_0]$$

$$P_1(x) = f(x) = f(x_0) + (x - x_0)f[x_1, x_0]$$

First degree interpolating polynomial

"Linear Interpolation"

May be found from Taylor's sereies

Example:

i	x_i	$f(x_i)$	$f[x_1, x_0]$
0	0	1	2
1	1	5	

a.

$$f[x_1, x_0] = \frac{f[x] - f[x_0]}{x_1 - x_0} = \frac{5-1}{3-1} = \frac{4}{2} = 2$$

$$f(x) = P_1(x) = f(x_0) + (x - x_0)f[x_1, x_0]$$

$$f(x) = 1 + (x - 1) * 2$$

$$f(x) = 2x - 1 \Longrightarrow \boxed{P_1(x) = 2x - 1}$$

Which is the straight line first-degree polynomial pass through the point (1,1) and (3,5).

b. Find $f(x)$ when $x = 0.5$

$$P_1(0.5) = 2 * 0.5 - 1 = 1 - 1 = 0$$

i	x_i	$f(x_i)$
0	1	1
	0.5	0
1	3	5

Quadratic Interpolation and Higher-Degree Interpolation

$$P_2(x) = a_0 + a_{1x} + a_2 x^2$$

Or, the second degree interpolating polynomial pass through $\{(x_0, f(x_0)), (x_1, f(x_1)), \& \ (x_2, f(x_2)\}$ is:

$$\begin{aligned} P_2(x) = f(x_0) + (x - x_0)f[x_1, x_0] \\ + (x - x_0)(x - x_1)f[x_2, x_1, x_0] \end{aligned}$$

- The third degree polynomial $f(x) = P_3(x)$

$$\begin{aligned} P_2(x) = f(x_0) + (x - x_0)f[x_1, x_0] \\ + (x - x_0)(x - x_1)f[x_2, x_1, x_0] \\ + (x - x_0)(x - x_1)(x - x_3)f[x_3, x_2, x_1, x_0] \end{aligned}$$

3rd degree polynomial pass through the points $(x_0, f(x_0))$, $(x_1, f(x_1))$ $(x_2, f(x_2)), and (x_3, f(x_3))$.

- nth –degree divided differences interpolating polynomial $P_n(x)$ has the form:

$$\begin{aligned} P_2(x) = f(x_0) + (x - x_0)f[x_1, x_0] \\ + (x - x_0)(x - x_1)f[x_2, x_1, x_0] + \cdots \\ + (x - x_0)(x - x_1)(x - x_3) + \cdots \\ + (x - x_{n-1})f[x_n, x_{\text{n}-1}, x_{n-2}, \dots, x_0] \end{aligned}$$

With reminder term $R(x)$, then

$$\boxed{f(x) = P_n(x) + R_n(x)}$$

Example:

				1st		2nd		3rd
i	x_i	$f(x_i)$		$f_1[x_1,x_0]$		$f_2[x_2,x_1,x_0]$		$f_3[x_3,x_2,x_1,x_0]$
0	0	-5						
			$\longrightarrow$	6				
1	1	1			$\longrightarrow$	2		
			$\longrightarrow$	12			$\rightarrow$	1
2	3	25			$\longrightarrow$	6		
			$\longrightarrow$	30				
3	4	55						

Points $(0,-5),(1,1),(3.25),(4,55) \Longrightarrow$ 3rd degree interpolating polynomial.

<u>Solution</u>:

$$f[x_1,x_0], f[x_2,x_1], f[x_3,x_2]$$

$$1.\ f[x_1,x_0] = \frac{f(x_1)-f(x_0)}{(x_1-x_0)} = \frac{f(x_1)}{x_1-x_0} - \frac{f(x_0)}{x_1-x_0}$$

$$= \frac{f(x_0)}{x_0-x_1} + \frac{f(x_1)}{x_1-x_0}$$

$$\therefore f[x_1,x_0] = \frac{-5}{0-1} + \frac{1}{1} = 5+1 = 6$$

$$2.\ f[x_2,x_1] = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{f(x_1)}{x_1-x_2} + \frac{f(x_2)}{x_2-x_1}$$

$$= \frac{1}{1-3} + \frac{25}{3-1} = \frac{1}{-2} + \frac{25}{2} = \frac{24}{2} = 12$$

$$3.\ f[x_3,x_2] = \frac{f(x_3)-f(x_2)}{x_3-x_2} = \frac{f(x_2)}{x_2-x_3} + \frac{f(x_3)}{x_3-x_2}$$

$$= \frac{25}{3-4} + \frac{55}{4-3} = \frac{25}{-2} + \frac{55}{2} = 30$$

2nd-divided difference,

$$f[x_2,x_1,x_0] = \frac{f[x_2,x_1] - f[x_1,x_0]}{x_2 - x_0}$$

$$= \frac{1}{(x_2 - x_0)} f[x_2,x_1] - \frac{1}{(x_2 - x_0)} f[x_1,x_0]$$

$$f[x_2,x_1,x_0] = \frac{1}{x_2 - x_0} . \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{1}{(x_2 - x_0)} . \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}$$

$$= \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_2)}{(x_2 - x_0)(x_1 - x_0)} + \frac{f(x_0)}{(x_2 - x_0)(x_1 - x_0)}$$

$$P_2(x) = f(x_0) + (x - x_0)f[x_1,x_0] + (x - x_0)(x - x_1)f[x_2,x_1,x_0]$$

$$P_2(x) = -5 + (x - 0)(6) + (x - 0)(x - 1)(2)$$

$$\boxed{P_2(x) = -5 + 6x + 2x^2 - 2x}$$

* Find $P_2(0.5)$?

$$P_2(0.5) = -5 + 6(0.5) + 2(0.5)^2 - 2(0.5)$$
$$= -5 + 3 + 0.5 - 1 = -2.5$$

$$\boxed{P_2(0.5) = -2.5}$$

$$f[x_3,x_2,x_1] = \frac{f[x_3,x_2,x_1] - f[x_2,x_1,x_0]}{x_3 - x_0} = 6$$

Three finite divided difference

$$f[x_3,x_2,x_1,x_0] = 1$$

$$P_3(x) = -5 + (x-0)(6) + (x-0)(x-1)(2) + (x-0)(x-3)(x-3)(1)$$

$$P_3(x) = (0.5)^3 - 2(0.5)^2 + 7(0.5) - 5$$

$$P_3(0.5) = (0.5)^3 - 2(0.5)^2 + 7(0.5) - 5$$

$$P_3(0.5) = \frac{1}{8} - \frac{1}{2} + \frac{7}{2} - 5 = \frac{-15}{8}$$

$$\Longrightarrow \boxed{P_3(0.5) = \frac{-15}{8}}$$

Homework:

Q1/ from the divided-difference table for the data in the table below:

i	x_i	$f(x_i)$
0	1.70	4.5624930
1	1.75	4.5974591
2	1.80	4.6321700
3	1.85	4.6666660
4	1.90	4.7009872
5	1.95	4.7351677
6	2.00	4.7692389

a. Estimate the magnitude of the error terms for linear interpolation using successive ordinate.

b. Given the additional information that tabulated function is:

$$f(x) = \cos(\ln x) + x + 2$$

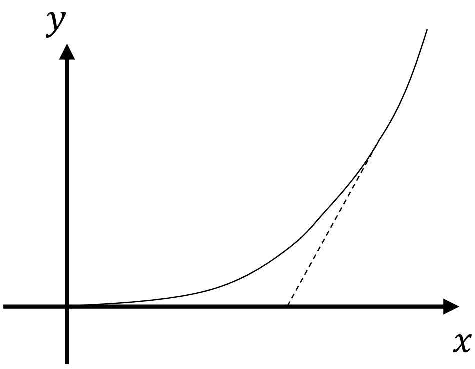
Show that the maximum error for any interpolant value approximating $f(x)$ using Linear interpolation with successive base point is smaller than 0.00004 (i.e. $\epsilon < 4 * 10^{-5}$)

Q2/ what are the main differences between:

Interpolation, Extrapolation, and inverse extrapolation. Then give an example.

Q3/ Suppose you have data pair follows the follows behavior:

Hint: if you extrapolate the tangent line,



a. How can you find the line-axis intercept?

b. How can one fixed data pair on the dashed line?

c. Can one use analytical, and/or numerical method? Explain How.

Question3/ Return back to "Curve Fitting" Example… calculate the 2nd-degree polynomial for the given set of data pairs.

i	x_i	y_i
0	0.05	0.956
⋮	⋮	⋮
⋮	⋮	⋮
9	0.98	0.242
10	1.17	0.104

Use "Interpolation" then compare with curve fitting solution.

Q4/ Find the line that best fits the points: $(-3,2), (1,-1)$ and $(2,0)$ in the least squares sense, i.e., the line that minimizes the mean square error between it and the points.

Q5/ The points $(2,2), (1,2)$, and $(3, B)$ are interpolated by the Newton polynomial:

$$P_2(x) = 2 + A(x-2) - (x-2)(x-1)$$

Find A & B.