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DERIVATIONS ON SEMIPRIME RINGS

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Abstract: In this paper we investigate some properties of derivations on prime and semiprime rings. Among other results we prove that if R is a semiprime ring, I is a nonzero two-sided ideal of R and f, g are derivations of R satisfying $f(x)y+yg(x) = 0$ for all $x, y \in I$, then $f(u)[x, y] = [x, y]g(u) = 0$ for all $x, y \in I$; in particular, f and g map I into the center of R. If R is a noncommutative prime ring, then $f = g = 0$ on R, which may be regarded as an analog of Posner Lemma for a pair of derivations satisfying this identity.

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1. Introduction

Throughout, R denotes an associative ring with center $Z(R)$. We write the commutator $[x, y] = xy - yx$ for $x, y \in R$. We shall frequently use the commutator identities $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$ for all $x, y, z \in R$. We recall that R is *prime* if $aRb = (0)$ implies $a = 0$ or $b = 0$; it is *semiprime* if $aRa = (0)$ implies $a = 0$. An additive map $d : R \to R$ is called a

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derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A mapping $f : R \to R$ is called *centralizing* if $[f(x), x] \in Z(R)$; in particular, if $[f(x), x] = 0$ for all $x \in R$, then it is called *commuting*. There has been considerable interest in commuting and centralizing derivations and related maps on prime and semiprime rings. A classical result in the theory of centralizing derivations is a Theorem of E. Posner [6, Theorem 2] which states that noncommutative prime rings do not admit nonzero centralizing derivations. A number of algebraists have extended Posner's result in several ways (see e.g. [2, 4, 5, 9], where further references are given).

Our aim in this paper is to present some results, which can be regarded as a contribution to the theory of derivations on prime and semiprime rings. Our work is inspired by Vukman [9]. For instance, Vukman [9] has proved that if f is a derivation on a noncommutative prime ring R of characteristic not two such that the mapping $x \to [d(x), x]$ is commuting on R, then $d = 0$. Alternatively, this result states that if d is a derivation, which satisfies the identity $d(x)x^2 + x^2d(x) - 2xd(x)x = 0$ for all $x \in R$, then $d = 0$. This is, in fact, an analog of Posner's result for derivations satisfying this identity. We work here with similar situations for a pair of derivations. For instance, we show (Theorem 2.2) that if R is a semiprime ring, I a nonzero two-sided ideal of R and f, g is a pair of derivations of R such that $f(x)y + yg(x) = 0$ for all $x, y \in I$, then $f(u)[x, y] = [x, y]g(u) = 0$ for all $x, y \in I$. Further, f and g map I into $Z(R)$. In particular, if R is a noncommutative prime ring, then $f = g = 0$ (Corollary 2.3). These results also improve earlier work of Thaheem [7, 8]. Further, some known facts follow as applications of the results here.

2. The Results

We shall need the following easy lemma in the proof of Theorem 2.2. We include it here for completeness.

Lemma 2.1. *Let R be a semiprime ring, I a nonzero two-sided ideal of R and $a \in R$ such that $axa = 0$ for all $x \in I$, then $a = 0$.*

Proof. Let $v \in R$. Then, by assumption, $avxa = 0$; that is, $xavxa = 0$ for all $v \in R$. By the semiprimeness of R, $xa = 0$ for all $x \in I$. Similarly, $ax = 0$ for all $x \in I$. Therefore, $ax = xa$ for all $x \in I$ and hence $a \in Z(I) \subseteq Z(R)$. So, by hypothesis, $a^3 = 0$. Since the center of R is free from nonzero nilpotents, therefore $a = 0$. □

We now prove our main result for semiprime rings. In case of prime rings, this is closely related to a result of Brešar [3].

Theorem 2.2. *Let R be a semiprime ring, I a nonzero two-sided ideal of R and f, g be derivations of R such that*

$$f(x)y + yg(x) = 0 \quad \text{for all } x, y \in I. \tag{1}$$

Then $f(u)[x, y] = [x, y]g(u) = 0$ for all $x, y, u \in I$; in particular, f and g map I into $Z(R)$.

Proof. Replacing x by yx in (1), we get

$$\begin{aligned} f(yx)y + yg(yx) &= f(y)xy + yf(x)y + yg(y)x + y^2g(x) \\ &= (f(y)xy + yg(y)x) + y(f(x)y + yg(x)) \\ &= f(y)xy + yg(y)x = 0. \end{aligned}$$

That is,

$$f(y)xy + yg(y)x = 0 \quad \text{for all } x, y \in I. \tag{2}$$

By (1), $yg(y) = -f(y)y$. So from (2), we get

$$f(y)xy - f(y)yx = f(y)(xy - yx) = f(y)[x, y] = -f(y)[y, x] = 0.$$

That is,

$$f(y)[y, x] = 0 \quad \text{for all } x, y \in I. \tag{3}$$

Let $w \in I$. Replacing x by wx in (3), we get $f(y)[y, wx] = f(y)w[y, x] + f(y)[y, w]x = 0$ and by (3), we get

$$f(y)w[y, x] = 0 \quad \text{for all } x, y, w \in I. \tag{4}$$

Linearizing (3) (in y) and using (3), we get

$$\begin{aligned} f(y + u)[y + u, x] &= (f(y) + f(u))([y, x] + [u, x]) \\ &= (f(y)[y, x] + f(u)[u, x]) + f(y)[u, x] + f(u)[y, x] \\ &= f(y)[u, x] + f(u)[y, x] = 0. \end{aligned}$$

That is, $f(y)[u, x] = -f(u)[y, x]$. So,

$$f(y)[u, x] = f(u)[x, y] \quad \text{for all } x, y, u \in I. \tag{5}$$

We want to prove that $f(u)[x, y] = 0$. For this purpose, let $v \in R$ and consider $f(u)[x, y]vf(u)[x, y]$. Then by (5), we have

$$f(u)[x, y]vf(u)[x, y] = f(u)[x, y]vf(y)[u, x] \quad \text{for all } x, y, u \in I \text{ and all } v \in R. \tag{6}$$

Put $w = [x, y]vf(y)$. As $[x, y] \in I$ and I is an ideal, so $w \in I$. Therefore, by (4) and (6), we get $f(u)[x, y]vf(u)[x, y] = f(u)w[u, x] = 0$ for all $v \in R$ and hence by the semiprimeness of R, $f(u)[x, y] = 0$ for all $x, y, u \in I$. That is

$$f(u)[x, y] = 0 \quad \text{for all} \;\; x, y, u \in I. \tag{7}$$

This proves the first identity.

We now show that $f(u) \in Z(I)$, where $Z(I)$ denotes the center of I. Replacing x by $xf(u)$ in (7) (and using (7)), we get

$$\begin{aligned} f(u)[xf(u), y] &= f(u)x[f(u), y] + f(u)[x, y]f(u) \\ &= f(u)x[f(u), y] = 0. \end{aligned}$$

So,

$$f(u)x[f(u), y] = 0 \quad \text{for all} \;\; x, y, u \in I. \tag{8}$$

Replacing x by yx in (8), we get

$$f(u)yx[f(u), y] = 0 \quad \text{for all} \;\; x, y, u \in I. \tag{9}$$

Multiplying (8) by $-y$ on the left, we get

$$-yf(u)x[f(u), y] = 0 \quad \text{for all} \;\; x, y, u \in I. \tag{10}$$

Adding (9) and (10), we get

$$(f(u)y - yf(u))x[f(u), y] = [f(u), y]x[f(u), y] = 0.$$

That is,

$$[f(u), y]x[f(u), y] = 0 \quad \text{for all} \;\; x, y, u \in I. \tag{11}$$

By Lemma 2.1, $[f(u), y] = 0$ for all $u, y \in I$.

This shows that $f(u) \in Z(I)$. Since u is an arbitrary element of I, we get $f(I) \subseteq Z(I) \subseteq Z(R)$. This proves another assertion about f.

We now show that $[x, y]g(u) = 0$ for all $x, y, u \in I$. Since $f(u) \in Z(R)$, therefore by (7), we have $f(u)[x, y] = [x, y]f(u) = 0$. So,

$$\begin{aligned} [x, y]f(u) &= xyf(u) - yxf(u) = xf(u)y - yf(u)x \\ &= -xyg(u) + yxg(u) = (yx - xy)g(u) \\ &= [y, x]g(u) = 0. \end{aligned}$$

That is,

$$[x, y]g(u) = 0 \quad \text{for all} \;\; x, y, u \in I. \tag{12}$$

This gives the second identity.

We now show that $g(I) \subseteq Z(I)$. Let $u \in I$. Replace x by $xg(u)$ in (12) (and using (12)), we get

$$\begin{aligned}[xg(u), y]g(u) &= [x, y]g(u)g(u) + x[g(u), y]g(u) \\ &= x[g(u), y]g(u) = 0.\end{aligned}$$

So,

$$[g(u), y]g(u)x[g(u), y]g(u) = 0 \quad \text{for all } u, y, x \in I. \tag{13}$$

From (13) and Lemma 2.1, we have $[g(u), y]g(u) = 0$. That is,

$$[g(u), y]g(u) = 0 \quad \text{for all } y, u \in I. \tag{14}$$

Replacing y by yv in (14) (and using (14)), we get

$$\begin{aligned}[g(u), yv]g(u) &= [g(u), y]vg(u) + y[g(u), v]g(u) \\ &= [g(u), y]vg(u) = 0.\end{aligned}$$

That is,

$$[g(u), y]vg(u) = 0 \quad \text{for all } u, y, v \in I. \tag{15}$$

Multiplying (15) by y on the right, we get

$$[g(u), y]vg(u)y = 0 \quad \text{for all } u, y, v \in I. \tag{16}$$

Also, replacing v by vy in (15), we get

$$[g(u), y]vyg(u) = 0 \quad \text{for all } u, y, v \in I. \tag{17}$$

Subtracting (17) from (16), we get

$$[g(u), y]v[g(u), y] = 0 \quad \text{for all } u, y, v \in I. \tag{18}$$

From (18) and Lemma 2.1, we get $[g(u), y] = 0$ for all $u, y \in I$. This implies that $g(u) \in Z(I)$; that is, $g(I) \subseteq Z(I) \subseteq Z(R)$. □

By Theorem 2.2, $f(x), g(x) \in Z(R)$ for all $x \in I$. Therefore, f and g are trivially centralizing on I and hence by Bell and Martindale [1, Theorem 4], $f = g = 0$. This establishes the following corollary.

Corollary 2.3. *Let R be a noncommutative prime ring, I a nonzero two-sided ideal of R and f, g be derivations of R such that*

$$f(x)y + yg(x) = 0 \quad \textit{for all } x, y \in I. \tag{19}$$

Then $f = g = 0$ *on* R.

Remark 2.4. (a) We shall make use of the following well-known results:

(i) Let R be a prime ring and I a nonzero two-sided ideal of R. Then I is a prime subring;

(ii) A noncommutative prime ring does not contain nonzero commutative left ideals.

(b) If R is a noncommutative prime ring, I a nonzero two-sided ideal of R such that $f(x)x = 0$ for all $x \in I$, then $f = 0$ on R. This follows from Corollary 2.3. Indeed, put $g = 0$ and $y = x$ in (19), we get $f(x)x = 0$ for all $x \in I$ and hence $f = 0$ on R.

Theorem 2.5. *Let* R *be a noncommutative prime ring,* I *a nonzero two-sided ideal of* R *and* f, g *be derivations of* R *such that*

$$f(x)xy + yg(x)x = 0 \quad \text{for all } x, y \in I. \tag{20}$$

Then $f = g = 0$.

Proof. Replacing x by $x + y$ in (20), we get

$$\begin{aligned} & f(x+y)(x+y)y + yg(x+y)(x+y) \\ &\quad = (f(x)+f(y))(x+y)y + y(g(x)+g(y))(x+y) \\ &\quad = (f(x)+f(y))(xy+y^2) + (yg(x)+yg(y))(x+y) \\ &\quad = (f(x)xy + yg(x)x) + (f(y)y^2 + yg(y)y) + f(x)y^2 \\ &\qquad + f(y)xy + yg(x)y + yg(y)x = 0. \end{aligned}$$

By (20), we get

$$(f(x)y + f(y)x)y + y(g(x)y + g(y)x) = 0 \quad \text{for all } x, y \in I. \tag{21}$$

Replacing x by xy in (21), we get

$$\begin{aligned} & (f(xy)y + f(y)(xy))y + y(g(xy)y + g(y)xy) \\ &\quad = (f(x)y^2 + xf(y)y + f(y)xy)y + y(g(x)y^2 + xg(y)y + g(y)xy) \\ &\quad = f(x)y^3 + xf(y)y^2 + f(y)xy^2 + yg(x)y^2 + yxg(y)y + yg(y)xy \\ &\quad = (f(x)y^3 + f(y)xy^2 + yg(x)y^2 + yg(y)xy) + (xf(y)y^2 + yxg(y)y) \\ &\quad = (f(x)y^2 + f(y)xy + yg(x)y + yg(y)x)y + (xf(y)y^2 + yxg(y)y) \\ &\quad = ((f(x)y + f(y)x)y + y(g(x)y + g(y)x))y \\ &\qquad + xf(y)y^2 + yxg(y)y = 0. \end{aligned} \tag{22}$$

By (21) and (22), we get

$$xf(y)y^2 + yxg(y)y = 0 \qquad \text{for all } \ x, y \in I. \tag{23}$$

Putting $x = y$ in (20), we get

$$f(y)y^2 + yg(y)y = 0 \qquad \text{for all } \ y \in I. \tag{24}$$

Multiplying (24) by x on the left, we get

$$xf(y)y^2 + xyg(y)y = 0 \quad \text{for all } \ x, y \in I. \tag{25}$$

Subtracting (25) from (23), we get

$$yxg(y)y - xyg(y)y = (yx - xy)g(y)y = [y, x]g(y)y = 0.$$

That is,

$$[x, y]g(y)y = 0 \quad \text{for all } \ x, y \in I. \tag{26}$$

Replacing x by xv in (26) (and using (26)), we get

$$[xv, y]g(y)y = x[v, y]g(y)y + [x, y]vg(y)y = [x, y]vg(y)y = 0.$$

That is,

$$[x, y]vg(y)y = 0 \ \text{ for all } \ x, y, v \in I. \tag{27}$$

Since I is noncommutative (Remark 2.4 (a)), therefore identity (27) together with the primeness of I (Remark 2.4(a)) implies that $g(y)y = 0$ for all $y \in I$. Then $g = 0$ on R (Remark 2.4(b)).

We now prove that $f = 0$. Since $g = 0$, therefore by (20), $f(x)xy = 0$ for all $x, y \in I$. Replacing y by vy, we get $f(x)xvy = 0$ for all $v \in I$. Since $I \neq (0)$ and R is prime, we get $f(x)x = 0$ for all $x \in I$ and hence $f = 0$ on R. □

We conclude the paper with following remark.

Remark 2.6. A mapping f of a ring R is called *skew-commuting* on a subset S of R if for any $x \in S$, $f(x)x + xf(x) = 0$; it is called *semi-commuting* on S if for any $x \in S$, either $f(x)x + xf(x) = 0$ or $f(x)x - xf(x) = 0$. If we substitute $y = x$ and $g = f$ in Corollary 2.3, then we have:

(a) Let R be a noncommutative prime ring, I a nonzero ideal of R and f a derivation of R which is skew-commuting on I, then $f = 0$ on R.

If we substitute $y = x$ and $g = -f$ in Corollary 2.3, then by (a) above, we have:

(b) Let R be a noncommutative prime ring, I a nonzero ideal of R and f a derivation of R which is semi-commuting on I, then $f = 0$ on R.

(c) We observe that (a) and (b) are analogs of Posner Theorem [6, Lemma 3] for skew-commuting and semi-commuting derivations, respectively.

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