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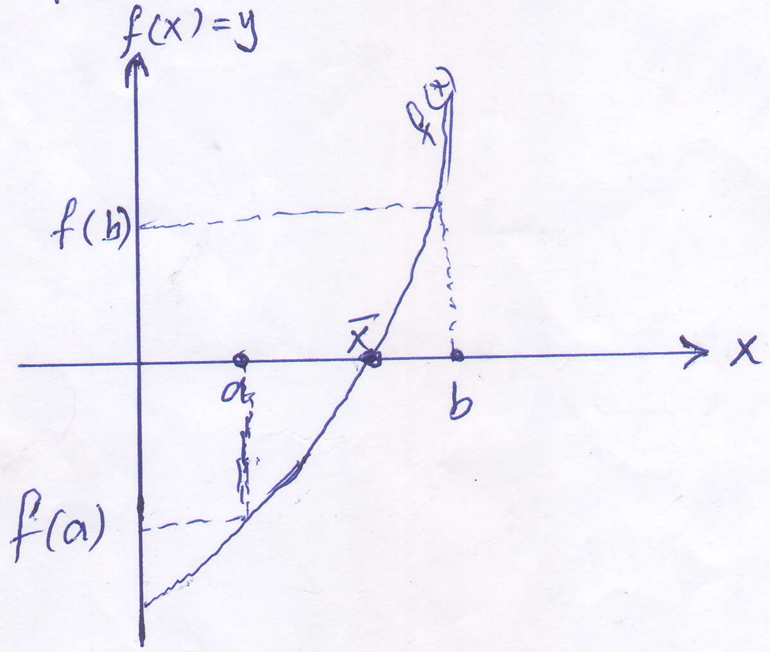
Numerical methods :

1) Bisection method :-

- Let $f(x)$ be a continuous function of (x) on the interval $[a,b]$.
- The equation $f(x)=0$ has a root in the interval $[a,b]$ if the following relation holds :

$$f(a) * f(b) < 0$$

i.e. $f(a)$ and $f(b)$ have different signs.



Steps of Algorithm :-

1. Choose an interval $[a,b]$ such that

$$f(a) * f(b) < 0$$

2. Find the value of (x_i) by dividing the distance between a and b : $x_i = \dfrac{a+b}{2}$

3- Calculate $f(x_i)$ (by using x_i-value).

4- If $f(a) * f(x_i) < 0 \Rightarrow b = x_i$

If $f(a) * f(x_i) > 0 \Rightarrow a = x_i$

5- Repeat the above procedure starting from step 2) to calculate a new (x_i) --- and so on.

6- Terminate the calculations when the given accuracy condition is satisfied.

We can use one of the following stopping conditions :-

ⓐ $|x_{i+1} - x_i| \leq \epsilon$

ⓑ $\left| \dfrac{x_{i+1} - x_i}{x_i} \right| \leq \epsilon$

ⓒ $|f(x_i)| \leq \epsilon$

ⓓ $\left| \dfrac{x_{i+1} - x_i}{x_{i+1}} \right| * 100 \leq \epsilon$

Example: Find the approximate root of the following equation by using bisection method:

$3x^2 = x + 5.25$ in the interval $[0.5, 3]$ and $\epsilon = 0.4$?

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Solution

$a = 0.5$, $b = 3$, $\epsilon = 0.4$, $f(x) = 3x^2 - x - 5.25$

$$x_i = \frac{a+b}{2}, \quad f(a) * f(x_i) < 0 \Rightarrow b = x_i$$
$$f(a) * f(x_i) > 0 \Rightarrow a = x_i$$

the condition of stopping is $\boxed{|f(x_i)| < 0.4}$

a	b	x_i	$f(a)$	$f(x_i)$	$f(a) * f(x_i)$
0.5	3	1.75	−5	2.1875	−
0.5	1.75	1.125	−5	−2.5781	+
1.125	1.75	1.4375	−2.5781	−0.4882	+
1.4375	1.75	1.5937	−0.4882	0.7759	−
.4375	1.5937	1.5156	−0.4882	0.1255	

$\therefore \bar{X} = 1.5156$ is a root.

Exercise Find the approximate root of the following equation by using bisection method? $2x - 1 = 0$ in the interval $[0, 2]$?

Example: Use bisection method to determine the positive root of the equation

$e^{-x} = x$ and $\epsilon = 6\%$?

Solution:

$f(x) = e^{-x} - x$.

To find the interval for (x):

x	0	1
$f(x)$	1	-0.63

$\Rightarrow \begin{cases} a = 0 \Rightarrow f(a) = 1 \\ b = 1 \Rightarrow f(b) = -0.63 \end{cases}$

$\therefore a = 0$ and $b = 1$

We have $f(a) * f(b) < b \Rightarrow$ there is a root in the interval $[0, 1]$.

i	a	b	x_i	$e\%$	$f(a)$	$f(x_i)$	$f(a) * f(x_i)$
1	0	1	0.5	---	+	+	+
2	0.5	1	0.75	$\simeq 33\%$	+	−	−
3	0.5	0.75	0.625	$\simeq 20\%$	+	−	−
4	0.5	0.625	0.5625	$\simeq 11\%$	+	+	+
5	0.5625	0.625	0.5937	$\simeq 5.2\%$	+	−	−

$\therefore$ the root is : $\bar{x} \simeq 0.5937$

③

Exercise: By the bisection method, find the approximate root for the following equations:

1) $\cos x = x^2$ in the interval $[0,1]$ and $\epsilon = 0.01$?

2) $\frac{1}{x} + 1 = 0$ in the interval $[-2, -0.5]$ and $\epsilon = 0.05$?