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Chapter 3

Solving systems of linear equations

A system of linear equations is a collection of two or more linear equations involving the same set of variables (unknowns)

* In a system of linear equations, we have:

No. of eqs. = No. of unknows.

The simplest kind of linear system involves two equations and two variables, For example:

$$2x + 3y = 6$$
$$4x + 9y = 15.$$

A system of three linear equations, for example:

$$6x + 4y - 2z = 20$$
$$x - 10y - 7z = 15$$
$$-x + 30y + z = -1$$

The general form of system of linear equations defined by:-

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n
\end{array}
$$

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Some properties of matrices :-

$AX = b$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

A is called coefficient matrix.
X is called unknowns vector.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ is called square matrix 3×3.

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$ is called upper triangular matrix 3×3

$$A_2 = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ is called lower triangular matrix 3×3

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$$A_3 = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$ is called diagonal matrix 3×3

$$A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is called identity matrix and denoted by I, 3×3.

$$A * A^{-1} = I.$$

There are two types of method to solve a system of linear equations :-

A- Direct methods.

B- Iterative methods.

A- Direct methods :-

1- Gaussian elimination :-

a- Forward Substitution :-

$$\left.\begin{aligned} a_{11}X_1 + a_{12}X_2 + a_{13}X_3 &= C_1 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 &= C_2 \\ a_{31}X_1 + a_{32}X_2 + a_{33}X_3 &= C_3 \end{aligned}\right\} \Longrightarrow$$

$$a''_{11}X_1 = C''_1 \quad \text{---} ①$$

$$a'_{21}X_1 + a'_{22}X_2 = C'_2 \quad \text{---} ②$$

$$a_{31}X_1 + a'_{32}X_2 + a_{33}X_3 = C_3 \quad \text{---} ③$$

④

Forward Substitution:

From ① ⟹ Find (x_1).

From ② ⟹ Find (x_2) {by using (x_1) from the previous step}.

From ③ ⟹ Find (x_3) {by using (x_1) and (x_2) from the previous steps}.

Or, we convert the coefficient matrix into a triangular form (leaving the lower elements)

u-elements

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Longrightarrow \begin{bmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & 0 \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

L-elements

Lower triangular matrix

Elimination produce

$*$ $a_{11}X_1 + a_{12}X_2 + a_{13}X_3 = C_1$

$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 = C_2$

$a_{31}X_1 + a_{32}X_2 + a_{33}X_3 = C_3$

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* we write the augmented matrix:

$$\begin{array}{l} \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right] \end{array} \begin{array}{ll} \text{Row 1} & (R_1) \\ \text{Row 2} & (R2) \\ \text{Row 3} & (R3) \end{array}$$

* To eliminate a_{13}: pivot row is ($R3$) and pivot element is a_{33}

New $R_1 = R_1 - R_3\left(\frac{a_{13}}{a_{33}}\right)$ we get:

$$\begin{bmatrix} a'_{11} & a'_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

* To eliminate a_{23}: pivot row is R_3 and pivot element is a_{33},

New $R_2 = R_2 - R_3\left(\frac{a_{23}}{a_{33}}\right)$, we get:

$$\left[\begin{array}{ccc|c} a'_{11} & a'_{12} & 0 & c'_1 \\ a'_{21} & a'_{22} & 0 & c'_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

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* To eliminate a'_{12} : Pivot Row is R_2, and pivot element is a'_{22}.

New $R_1 = R_1 - R_2\left(\frac{a'_{12}}{a'_{22}}\right)$, we get:

$$\begin{array}{c} \begin{bmatrix} a''_{11} & 0 & 0 \\ a'_{21} & a'_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

* New the set of eq.s is:

$$\begin{bmatrix} a''_{11} & 0 & 0 \\ a'_{21} & a'_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} C''_1 \\ C'_2 \\ C_3 \end{bmatrix}$$

or

$$a''_{11} X_1 = C''_1$$

$$a'_{21} X_1 + a'_{22} X_2 = C'_2$$

$$a_{31} X_1 + a_{32} X_2 + a_{33} X_3 = C_3$$

we can solve for X_1, X_2 and X_3 by forward substitution

B- Backward substitution :-

$$\left.\begin{aligned} a_{11} X_1 + a_{12} X_2 + a_{13} X_{13} &= C_1 \\ a_{21} X_1 + a_{22} X_2 + a_{23} X_3 &= C_2 \\ a_{31} X_1 + a_{32} X_2 + a_{33} X_3 &= C_3 \end{aligned}\right\} \Rightarrow$$

$$a_{11} X_1 + a_{12} X_2 + a_{13} X_3 = C_1 \quad \text{--- ①}$$

$$a'_{22} X_2 + a'_{23} X_3 = C'_2 \quad \text{--- ②}$$

$$a''_{33} X_3 = C''_3 \quad \text{--- ③}$$

From (3) $\Rightarrow$ find (x_3).

From (2) $\Rightarrow$ find (x_2) (using (x_3))

From (1) $\Rightarrow$ find (x_1) (using (x_3) and (x_2))

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

- Elimination procedure

* Augmented matrix:

$$\begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & c_1 \\ a_{21} & a_{22} & a_{23} & \vdots & c_2 \\ a_{31} & a_{32} & a_{33} & \vdots & c_3 \end{bmatrix} \end{array} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

* To eliminate a_{21} and a_{31}: pivot row is (R_1) and pivot element is (a_{11}):

* a_{21} elimination:

$$\text{New } R_2 = R_2 - R_1\left(\frac{a_{21}}{a_{11}}\right)$$

* a_{31} elimination:

$$\text{New } R_3 = R_3 - R_1\left(\frac{a_{31}}{a_{11}}\right)$$

* we get

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & c_1 \\ 0 & a'_{22} & a'_{23} & \vdots & c'_2 \\ 0 & a'_{32} & a'_{33} & \vdots & c'_3 \end{bmatrix}$$

* To eliminate a_{32} : Pivot row is R_2, and pivot element is a_{22}:

New $R_3 = R_3 - R_2 \left(\frac{a_{32}}{a_{22}}\right)$, we get:

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & c_1 \\ 0 & a'_{22} & a'_{23} & c'_2 \\ 0 & 0 & a''_{33} & c''_3 \end{array}\right]$$

* Now, the equations system is:

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 = C_1$$
$$a'_{22}X_2 + a'_{23}X_3 = C'_2$$
$$a''_{33}X_3 = C''_3$$

* We can solve for X_3, X_2 and X_1 by backward substitution.

Example 1

Use backward Gaussian elimination to solve the following system of linear equations:

$$100X_1 + 80X_2 - 40X_3 = 8$$
$$200X_1 - 40X_2 + 20X_3 = 6$$
$$300X_1 + 340X_2 - 100X_3 = -6$$

(9)

Sol.

Augmented matrix is :

$$\left[\begin{array}{ccc:c} 100 & 80 & -40 & 8 \\ 200 & -40 & 20 & 6 \\ 300 & 340 & -100 & -6 \end{array}\right] \begin{array}{l} R_1 \\ R_2 = R_2 - R_1\left(\frac{200}{100}\right) \\ R_3 = R_3 - R_1\left(\frac{300}{100}\right) \end{array}$$

$$\left[\begin{array}{ccc:c} 100 & 80 & -40 & 8 \\ 0 & -200 & 100 & -10 \\ 0 & 100 & 20 & -30 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 = R_3 - R_2\left(\frac{100}{-200}\right) \end{array}$$

$$\left[\begin{array}{ccc:c} 100 & 80 & -40 & 8 \\ 0 & -200 & 100 & -10 \\ 0 & 0 & 70 & -35 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

The equations system is :

$$100X_1 + 80X_2 - 40X_3 = 8 \quad --- (1)$$

$$-200X_2 + 100X_3 = -10 \quad --- (2)$$

$$70X_3 = -35 \quad --- (3)$$

* Using backward substitution :

From (3) : $X_3 = -35/70 \Rightarrow \boxed{X_3 = -0.5}$

From (2) : $-200X_2 = -10 - 100X_3$

$$X_2 = \frac{-10 - 100X_3}{-200} \Rightarrow X_2 = \frac{10 + 100(-0.5)}{200}$$

$$\boxed{X_2 = -0.2}$$

From ①: $100X_1 = 8 - 80X_2 + 40X_3$

$$X_1 = \frac{8 - 80(-0.2) + 40(-0.5)}{100}$$

$\Rightarrow \boxed{X_1 = 0.04}$

* The solution of the equations system are:

$X_1 = 0.04$, $X_2 = -0.2$, $X_3 = -0.5$

Exercise:

Use backward Gaussian elimination to solve the following system of linear equations:

$$3X_1 - X_2 + 2X_3 = 12$$
$$X_1 + 2X_2 + 3X_3 = 11$$
$$2X_1 - 2X_2 - X_3 = 2$$