CH3 : Limits , Continuity and Differentiation

S3.1 : Limits and Continuity

Remark 3.1.1: If the values of a function $y = f(x)$ can be made as close as
we like to a fixed number L by taking x close to x_0 (but not equal to x_0) we say that L is the limit of f as x approaches x_0 , and we write it as

$$\lim_{x \to x_0} f(x) = L$$

Also we can say that the limit of f as x approaches x_0 equals L .

Definition 3.1.2 :

Let f be a function defined on the set $(x_0 - p , x_0) \cup (x_0 , x_0 + p)$, with $p > 0$. Then

$$\lim_{x \to x_0} f(x) = L$$

iff for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

if $0 < |x - x_0| < \delta$ **then** $|f(x) - L| < \varepsilon$.

Theorem 1 :

1) $\lim_{x \to x_0} x = x_0$
2) $\lim_{x \to x_0} k = k$

Theorem 2 : If $\lim_{x \to x_0} f(x) = L_1$ and $\lim_{x \to x_0} g(x) = L_2$, then

1) $\lim_{x \to x_0} [f(x) + g(x)] = L_1 + L_2$
2) $\lim_{x \to x_0} [f(x) - g(x)] = L_1 - L_2$
3) $\lim_{x \to x_0} [f(x) \cdot g(x)] = L_1 \cdot L_2$
4) $\lim_{x \to x_0} [k \cdot f(x)] = k \cdot L_1$
5) $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{L_1}{L_2}$ if $L_2 \neq 0$.

Example 3.1.3 : Find each of the following :

1. $\lim\limits_{x \to -2} 7$
2. $\lim\limits_{x \to 1} x(3 - x)$
3. $\lim\limits_{x \to 3} (x^2 + 2x - 1)$
4. $\lim\limits_{x \to 2} \dfrac{x - 2}{x^2 - 5x + 6}$
5. $\lim\limits_{x \to 0} \dfrac{x^2 - 5x}{x}$

Solution :

1. $\lim\limits_{x \to -2} 7 = 7$
2. $\lim\limits_{x \to 1} x(3 - x) = 1(3 - 1) = 2$
3. $\lim\limits_{x \to 3} (x^2 + 2x - 1) = (3)^2 + 2(3) - 1 = 9 + 6 - 1 = 14$
4. $\lim\limits_{x \to 2} \dfrac{x - 2}{x^2 - 5x + 6} = \lim\limits_{x \to 2} \dfrac{x - 2}{(x - 3)(x - 2)} = \lim\limits_{x \to 2} \dfrac{1}{(x - 3)} = \dfrac{1}{2 - 3} = -1$
5. $\lim\limits_{x \to 0} \dfrac{x^2 - 5x}{x} = \lim\limits_{x \to 0} \dfrac{x(x - 5)}{x} = \lim\limits_{x \to 0} (x - 5) = 0 - 5 = -5$

Theorem 3 :

1) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

2) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

Example 3.1.4 : Find each of the following :

1. $\lim\limits_{x \to 0} \dfrac{\sin 4x}{\sin 5x}$
2. $\lim\limits_{x \to 0} \dfrac{3x}{\sin 2x}$

3. $\lim_{x\to 0} \frac{\tan x}{x}$

Solution :

1. $\lim_{x\to 0} \frac{\sin 4x}{\sin 5x} = \lim_{x\to 0} \frac{4x \cdot \frac{\sin 4x}{4x}}{5x \cdot \frac{\sin 5x}{5x}} = \frac{4}{5}$

2. $\lim_{x\to 0} \frac{3x}{\sin 2x} = \lim_{x\to 0} \frac{3x}{2x \cdot \frac{\sin 2x}{2x}} = \frac{3}{2}$

3. $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\frac{\sin x}{\cos x}}{\frac{x}{1}} = \lim_{x\to 0} \left(\frac{\sin x}{\cos x} \cdot \frac{1}{x}\right) = \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x}\right)$

$= 1 \times 1 \quad = 1$

Exercise 3.1.5 : Find each of the following :

1. $\lim_{x\to 0} \frac{1-\cos x}{x+\sin x}$
2. $\lim_{x\to\infty} (1+\cos\frac{1}{x})$
3. $\lim_{x\to 0} \frac{\sin 2x}{2x^2+x}$
4. $\lim_{y\to 0} \frac{\tan 2y}{3y}$
5. $\lim_{y\to\infty} \frac{y^4}{y^4-7y^3+3y^2+9}$

Definition 3.1.6 : A function $f(x)$ is said to be continuous at x_0 if

1) f is defined at x_0 (i.e. $f(x_0) = \mathrm{L}$ where $\mathrm{L} \in \mathrm{R}$).

2) $\lim_{x\to x_0} f(x)$ exists

3) $\lim_{x\to x_0} f(x) = f(x_0) = \mathrm{L}$

Example 3.1.7 : Let $f(x) = \begin{cases} x^2 & x \leq 1 \\ 3 - 2x & x > 1 \end{cases}$

Is f continuous at $x = 1$.

Solution :

1) $f(1) = 1^2 = 1$

2) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2 = 1^2 = 1$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (3 - 2x) = 3 - 2(1) = 1$

since $\lim\limits_{x \to 1^-} f(x) = 1 = \lim\limits_{x \to 1^+} f(x)$

Therefore $\lim\limits_{x \to 1} f(x)$ exists and $\lim\limits_{x \to 1} f(x) = 1$

3) $\lim\limits_{x \to 1} f(x) = 1 = f(1)$

Therefore f is continuous at $x = 1$

Example 3.1.8 : Let $f(x) = \begin{cases} 2x + 1 & if\ x < -2 \\ x^2 - 2 & if\ x \geq -2 \end{cases}$

Is f continuous at $x = -2$.

Solution :

1) $f(-2) = (-2)^2 - 2 = 4 - 2 = 2$.

2) $\lim\limits_{x \to (-2)^-} f(x) = \lim\limits_{x \to (-2)^-} (2x + 1) = 2(-2) + 1 = -4 + 1 = -3$

$\lim\limits_{x \to (-2)^+} f(x) = \lim\limits_{x \to (-2)^+} (x^2 - 2) = (-2)^2 - 2 = 4 - 2 = 2$

since $\lim\limits_{x \to (-2)^-} f(x) \neq \lim\limits_{x \to (-2)^+} f(x)$

Therefore $\lim\limits_{x \to (-2)} f(x)$ does not exists

Thus f is not continuous at $x = -2$.

Exercise 3.1.9 :

Let $f(x) = \begin{cases} \frac{x^2-2x-8}{x+2} & if \ x \neq -2 \\ -3 & if \ x = -2 \end{cases}$

Is f continuous at $x = -2$.

S3.2 : Differentiation

Definition of Derivative , Rules of Differentiation

Definition 3.2.1:

Let $y = f(x)$ be a function and let the variable x receive a certain increment Δx. Then the function y will receive a certain increment Δy. Thus for the value of x we have $y = f(x)$ and for the value of $x + \Delta x$, we have $y + \Delta y = f(x + \Delta x)$.

Thus the increment Δy is given by :

$$\Delta y = f(x + \Delta x) - f(x)$$

Remark 3.2.2 : **Δ is an abbreviation of difference (in x , y) and is not a factor .**

Forming the ratio of the increment of the function y to the increment of the variable x , we get

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is called the average rate of change of the function $y = f(x)$ with respect to the variable x . $\frac{\Delta y}{\Delta x}$ is also called the difference quotient of the function $y = f(x)$. If the limit of this ratio as Δx approaches zero exists, that is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

exist, then the function is called differentiable and the limit ($\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$) is called the first derivative of the function $y = f(x)$ with respect to