CH4 : Integration

S4.1 : The Indefinite Integral

Definition : A function $F(x)$ is anti-derivative of a function $f(x)$ with respect to x if $\frac{d}{dx}F(x) = f(x)$ for all x in the domain of f. The set of all anti derivatives of f is the indefinite integral of f with respect to x, denoted by $\int f(x)\,dx$ *i.e.* $\int f(x)\,dx = F(x) + c$.

The symbol $\int$ is an integral sign .

The function f is the integrand of the integral and x is the variable of the integration .

Example 4.1.1: $\int 3x^2\,dx = x^3 + c$.

Integral Formulas :

1) $\int u^n\,du = \frac{u^{n+1}}{n+1} + c$, $n \neq -1, n$ *rational*

$\int du = \int 1\,du = u + c$ (*special case*)

2) $\int \sin u\,du = -\cos u + c$

3) $\int \cos u\,du = \sin u + c$

4) $\int \sec^2 u\,du = \tan u + c$

5) $\int \csc^2 u\,du = -\cot u + c$

6) $\int \sec u \tan u\,du = \sec u + c$

7) $\int \csc u \cot u\,du = -\csc u + c$

8) $\int \frac{1}{u}\,du = \ln|u| + c$

9) $\int e^u\,du = e^u + c$

10) $\int a^u\,du = \frac{a^u}{\ln a} + c$, $a > 0$

S4.2 : The Definite Integral

Definition : If f is a continuous at every point of $[a,b]$ and if F is any anti-derivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

is called the definite integral .

Example 4.2.1 : Evaluate the integral $\int_1^4 (x^3+2x+9)\,dx$

Solution : $\int_1^4 (x^3+2x+9)\,dx = \left[\frac{x^4}{4}+x^2+9x\right]_1^4$

$$= \left(\frac{256}{4}+16+36\right) - \left(\frac{1}{4}+1+9\right)$$

$$= 116 - 10.25 = 105.75 .$$

Example 4.2.2 : Evaluate $\int_0^{\frac{\pi}{2}} \sin x\,dx$

Solution : $\int_0^{\frac{\pi}{2}} \sin x\,dx = [-\cos x]_0^{\frac{\pi}{2}} = 0-(-1) = 1$

How to Find the Area :

To find the area between the graph of $y = f(x)$ and the $x-axis$ over the interval $[a,b]$ we should follow the following steps :

Step 1 : Partition $[a,b]$ with the zeros of f .

Step 2 : Integrate f over each subinterval .

Step 3 : Add the absolute values of the Integrals .

Example 4.2.3 : Find the total area of the region between the curve $y = x^2+2x$ and the $x-axis$ over the interval $[-3,4]$.

Solution : $x^2+2x = 0 \Rightarrow x(x+2) = 0$

$\Rightarrow x = 0$ or $x = -2$

$$\therefore \text{ the area } = \left|\int_{-3}^{-2}(x^2+2x)\,dx\right| + \left|\int_{-2}^{0}(x^2+2x)\,dx\right| + \left|\int_{0}^{4}(x^2+2x)\,dx\right|$$

$$= \left|\left[\frac{x^3}{3}+x^2\right]_{-3}^{-2}\right| + \left|\left[\frac{x^3}{3}+x^2\right]_{-2}^{0}\right| + \left|\left[\frac{x^3}{3}+x^2\right]_{0}^{4}\right|$$

$$= \left|\left(-\frac{8}{3}+4\right)-\left(-\frac{27}{3}+9\right)\right|+\left|(0+0)-\left(-\frac{8}{3}+4\right)\right|$$

$$+\left|\left(\frac{64}{3}+16\right)-(0+0)\right|$$

$$= \frac{4}{3}+\frac{4}{3}+\frac{112}{3} = \frac{120}{3} = 40 .$$

Example 4.2.4 : Find the total area of the region between the curve $y = x^3 - 4x^2 + 3x$ and the $x - axis$ over the interval $[0,2]$.

Solution : $x^3 - 4x^2 + 3x = 0 \Rightarrow x(x^2 - 4x + 3) = 0$
$\Rightarrow x(x-1)(x-3) = 0 \Rightarrow x = 0\ (neglected)\ or\ x = 1$
$or\ x = 3\ (neglected)$

$$\therefore\ the\ area = \left|\int_0^1 (x^3 - 4x^2 + 3x)\,dx\right| + \left|\int_1^2 (x^3 - 4x^2 + 3x)\,dx\right|$$

$$= \left|\left[\frac{x^4}{4}-\frac{4x^3}{3}+\frac{3x^2}{2}\right]_0^1\right| + \left|\left[\frac{x^4}{4}-\frac{4x^3}{3}+\frac{3x^2}{2}\right]_1^2\right|$$

$$= \left|\left(\frac{1}{4}-\frac{4}{3}+\frac{3}{2}\right)-0\right| + \left|\left(\frac{16}{4}-\frac{32}{3}+\frac{12}{2}\right)-\left(\frac{1}{4}-\frac{4}{3}+\frac{3}{2}\right)\right|$$

$$= \left|\frac{3-16+18}{12}\right| + \left|\frac{48-128+72}{12} - \frac{3-16+18}{12}\right|$$

$$= \left|\frac{5}{12}\right| + \left|-\frac{8}{12}-\frac{5}{12}\right| = \left|\frac{5}{12}\right| + \left|-\frac{13}{12}\right| = \frac{5}{12}+\frac{13}{12} = \frac{18}{12} = 1.5$$

How to Find the Area Between Two Curves over an Interval [a , b]:

To find the area between the two curves $f(x)$ and $g(x)$ over the interval $[a,b]$ we should follow the following steps :

Step 1 : Partition $[a,b]$ with the zeros of $f - g$.

Step 2 : Integrate $f - g$ over each subinterval .

Step 3 : Add the absolute values of the Integrals .

Example 4.2.5 : Find the total area of the region between the two curves $f(x) = x^2$ and $g(x) = 2x$ over the interval $[-1,2]$.

Solution : $f(x) - g(x) = x^2 - 2x = 0 \Rightarrow x(x-2) = 0$

$x = 0\ \ or\ x = 2\ (neglected)$

$\therefore$ *the area* $= \left|\int_{-1}^{0}(x^2 - 2x)\, dx\right| + \left|\int_{0}^{2}(x^2 - 2x)\, dx\right|$

$= \left|\left[\frac{x^3}{3} - x^2\right]_{-1}^{0}\right| + \left|\left[\frac{x^3}{3} - x^2\right]_{0}^{2}\right|$

$= \left|(0-0) - \left(-\frac{1}{3} - 1\right)\right| + \left|\left(\frac{8}{3} - 4\right) - (0-0)\right|$

$= \left|+\frac{4}{3}\right| + \left|-\frac{4}{3}\right| = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

How to Find the Area Between Two Curves :

To find the area between the two curves $f(x)$ *and* $g(x)$ we should follow the following steps :

Step 1 : Find the zeros of $f - g$, and let them be a *and* b .

Step 2 : Integrate $f - g$ over the interval $[a, b]$.

Step 3 : Find the absolute value of the Integration found in step 2 .

Example 4.2.6 : Find the area of the region enclosed by the parabola $y = x^2 - 2$ *and the line* $y = x$.

Solution : $f(x) - g(x) = (x^2 - 2) - x = 0 \Rightarrow x^2 - 2 - x = 0$

$\Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2$ *or* $x = -1$.

$\therefore$ *the area* $= \left|\int_{-1}^{2}(x^2 - 2 - x)\, dx\right|$

$= \left|\left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2}\right|$

$= \left|\left(\frac{8}{3} - \frac{4}{2} - 4\right) - \left(-\frac{1}{3} - \frac{1}{2} + 2\right)\right|$

$= \left|\frac{8-6-12}{3} - \frac{-2-3+12}{6}\right|$

$= \left|-\frac{10}{3} - \frac{7}{6}\right| = \left|\frac{-20-7}{6}\right| + \left|-\frac{27}{6}\right| = \frac{27}{6} = 4\,\frac{1}{2}$.

Example 4.2.7 : Find the total area of the region enclosed by the parabola $f(x) = x^2$ *and the line* $g(x) = 2x$.

Solution : $f(x) - g(x) = x^2 - 2x = 0 \Rightarrow x(x-2) = 0$

$\Rightarrow x = 0$ *or* $x = 2$

$\therefore$ *the area* $= \left| \int_0^2 (x^2 - 2x) \, dx \right|$

$$= \left| \left[\frac{x^3}{3} - x^2 \right]_0^2 \right|$$

$$= \left| \left(\frac{8}{3} - 4 \right) - (0 - 0) \right|$$

$$= \left| \frac{8-12}{3} \right| = \left| -\frac{4}{3} \right| = \frac{4}{3}$$

Rules for definite Integrals

1) Order of integration :

$$\int_b^a f(x) \, dx = -\int_a^b f(x) \, dx$$

2) Zero integration :

$$\int_a^a f(x) \, dx = 0$$

3) Constant multiple :

$$\int_a^b k \, f(x) \, dx = k \int_a^b f(x) \, dx \quad \forall \; k \in R \text{ , and thus}$$

$$\int_a^b - f(x) \, dx = -\int_a^b f(x) \, dx \quad for \;\; k = -1 \; ..$$

4) Sum and difference :

$$\int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$$

5) Additively :

$$\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$$

Example 4.2.8 : Suppose that

$\int_{-2}^{1} f(x) \, dx = 4$, $\int_1^3 f(x) \, dx = -3$ and $\int_{-2}^{1} h(x) \, dx = 6$. Find

1) $\int_3^1 f(x) \, dx$

2) $\int_{-2}^{1} (2 f(x) + 5 \, h(x)) dx$

3) $\int_{-2}^{3} f(x)\,dx$

4) $\int_{-2}^{1} (3f(x) - 2h(x))dx$

Solution :

1) $\int_{3}^{1} f(x)\,dx = -\int_{1}^{3} f(x)\,dx = -(-3) = 3$.

2) $\int_{-2}^{1} (2f(x) + 5h(x))dx = \int_{-2}^{1} 2f(x)\,dx + \int_{-2}^{1} 5h(x)\,dx$

$= 2\int_{-2}^{1} f(x)\,dx + 5\int_{-2}^{1} h(x)\,dx$

$= 2(4) + 5(6) = 8 + 30 = 38$.

3) $\int_{-2}^{3} f(x)\,dx = \int_{-2}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx$

$= 4 + (-3) = 1$.

4) $\int_{-2}^{1} (3f(x) - 2h(x))dx = \int_{-2}^{1} 3f(x)\,dx - \int_{-2}^{1} 2h(x)\,dx$

$= 3\int_{-2}^{1} f(x)\,dx - 2\int_{-2}^{1} h(x)\,dx$

$= 3(4) - 2(6) = 12 - 12 = 0$.

Exercise 4.2.9 : Evaluate the following integrals :

1) $\int_{-2}^{0} (2x + 5)\,dx$

2) $\int_{0}^{1} \left(x^2 + \sqrt{x}\right) dx$

3) $\int_{0}^{\pi} (1 + \cos x)\,dx$

4) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (8y^2 + \sin y)\,dy$

5) $\int_{2}^{1} \frac{2}{x^2}\,dx$