(1) 2, 5, 8, 11, ...

(2) 5, 9, 13, 17, ...

(3) −5, 6, 17, 28, ...

S 5.2 : Arithmetic Series and Geometric Series

Definition 5.2.1 : If $\{a_n\}$ is an arithmetic sequence, then the corresponding series $\sum_{i=1}^{\infty} a_i$ is called an arithmetic series and the terms

$S_n = \sum_{i=1}^{n} a_i$ is called the nth partial sum of the arithmetic series.

Theorem (1) :

(i) The nth partial sum of an arithmetic series is

$$S_n = \frac{n}{2}(a_1 + a_n).$$

(ii) The nth partial sum of an arithmetic series is

$$S_n = n a_1 + \frac{n(n-1)}{2} d.$$

Example 5.2.2 : Find the sum of the first 100 positive integers.

Solution : $a_1 = 1$ and $a_{100} = 100$

$\therefore S_{100} = \frac{100}{2}(1 + 100) = 50(101) = 5050.$

Example 5.2.3 : Find the sum of the first 20 terms of the arithmetic sequence (the 20th partial sum of the arithmetic series) 10, 16, 22 ...

Solution :

$a_1 = 10$ and $d = 16 - 10 = 6$

$$\therefore S_{20} = 20 \times 10 + \frac{20(20-1)}{2} \times 6 = 200 + \frac{20(19)}{2} \times 6$$
$$= 200 + 190 \times 6$$
$$= 200 + 1140 = 1340 .$$

Example 5.2.4 : The sum of the first 16 terms of an arithmetic sequence is 80 . If $a_{16} = 20$, find a_1 and d.

Solution :

$$S_{16} = \frac{16}{2}(a_1 + a_{16}) = 8(a_1 + 20) = 8a_1 + 160$$
$$\therefore 80 = 8a_1 + 160 \Rightarrow 8a_1 = 80 - 160$$
$$\Rightarrow 8a_1 = -80$$
$$\Rightarrow a_1 = \frac{-80}{8} = -10$$

Since $a_{16} = a_1 + (16-1)d$ then $20 = -10 + 15d$

$$\Rightarrow 15d = 20 + 10 \Rightarrow 15d = 30 \Rightarrow d = \frac{30}{15} = 2 .$$

Example 5.2.5 : Find the sum of the first 12 terms of the sequence 11 , 18 , 25 , ...

Solution : $a_1 = 11$ and $d = 18 - 11 = 7$.

The ith term $a_i = a_1 + (i-1)d = 11 + (i-1) \times 7$

$$= 11 + 7i - 7 = 7i + 4$$

$$\therefore S_n = \sum_{i=1}^{n} (7i + 4)$$

$$\therefore S_{12} = \sum_{i=1}^{12} (7i + 4) = \frac{12}{2}(a_1 + a_{12}) = 6(11 + 88)$$
$$= 6 \times 99 = 594$$

Exercise 5.2.6 : Find each of the following sums :

(1) $\sum_{i=1}^{20} (3i-10)$ (3) $\sum_{i=1}^{25} (i+9)$

(2) $\sum_{i=1}^{15} (6i+5)$ (4) $\sum_{i=1}^{13} (2i+3)$

<u>Definition 5.2.7</u> : A sequence of numbers in which each term after the first term is obtained by multiplying the previous term by a fixed nonzero real number is called a geometric sequence.
The fixed nonzero real number that is multiplied is called the common ratio and will be denoted by r.

<u>Example 5.2.8</u> : $3, 6, 12, 24, 48, \ldots$ is a geometric sequence, since each term after the first term is obtained by multiplying the previous term by 2.
In this example we have $r = 2$ and

$a_1 = 3$

$a_2 = r \cdot a_1 = 2(3) = 6$

$a_3 = r \cdot a_2 = 2(6) = 12$

$a_4 = r \cdot a_3 = 2(12) = 24$

$\vdots$

<u>Example 5.2.9</u>: Find the common ratio r for the following geometric sequence

$7, 21, 63, 189, \ldots$

<u>Solution</u> : $r = \frac{21}{7} = 3$

or $r = \frac{63}{21} = 3$

or $r = \frac{189}{63} = 3$

66

Example 5.2.10 : Write the first five terms of the geometric sequence whose first term $a_1 = 3$ and common ratio $r = -2$.

Solution : The first term $a_1 = 3$

The second term $a_2 = r \cdot a_1 = (-2) \times 3 = -6$

The third term $a_3 = r \cdot a_2 = (-2)(-6) = 12$

The fourth term $a_4 = r \cdot a_3 = (-2) \times 12 = -24$

The fifth term $a_5 = r \cdot a_4 = (-2)(-24) = 48$.

Exercise 5.2.11 : Write the first seven terms of each of the following geometric sequence if you know that

(1) $a_1 = -6$ and $r = 2$

(2) $a_1 = 7$ and $r = -1$

(3) $a_1 = 4$ and $r = 5$

Remark 5.2.12 : In the geometric sequence with the first term a_1 and the common ratio r, the nth term a_n is given by $a_n = r^{n-1} \times a_1$

Example 5.2.13 : Find a_5, a_7 and a_n for the geometric sequence $5, 10, 20, \ldots$

Solution :

$a_1 = 5$ and $r = \frac{10}{5} = 2$

$\therefore a_5 = 2^{5-1} \times 5 = 2^4 \times 5 = 16 \times 5 = 80$

$a_7 = 2^{7-1} \times 5 = 2^6 \times 5 = 64 \times 5 = 320$

$a_n = 2^{n-1} \times 5$

(67) **Exercise 5.2.14 :** For each of the following geometric sequences, find r, a_6 and a_n.

(1) $6, 18, 54, \ldots$

(2) $7, -7, 7, \ldots$

(3) $-2, 4, -8, \ldots$

-2

Definition 5.2.15 : If $\{a_n\}$ is a geometric sequence, then the corresponding series $\sum_{i=1}^{\infty} a_i$ is called a geometric series and the terms

$$S_n = \sum_{i=1}^{n} a_i = a_1 + r a_1 + r^2 a_1 + \cdots + r^{n-1} a_1$$ is called

the nth partial sum of the geometric series.

Theorem (2) :

(i) The nth partial sum of a geometric series is

$$S_n = \begin{cases} \dfrac{a_1(1-r^n)}{1-r} & \text{if } r \neq 1 \\ n a_1 & \text{if } r = 1 \end{cases}$$

(ii) The nth partial sum of a geometric series is

$$S_n = \begin{cases} \dfrac{a_1 - r a_n}{1-r} & \text{if } r \neq 1 \\ n a_1 & \text{if } r = 1 \end{cases}$$

Example 5.2.16 : Find the sum S_7 of the first seven terms of the geometric sequence (the 7th partial sum of