(2) Gauss - Jordan Elimination Method:

To reduce the augmented matrix to reduced row - echelon form you should follow the following steps:

Step 1. Locate the leftmost column that does not consist entirely of zeros.

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

Step 3. If the entry that is now at the top of the column found in Step 1 is b , multiply the first row by $\frac{1}{b}$ in order to introduce a leading 1.

Step 4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

Step 5. Now cover the top row in the matrix and begin again with Step 1 applied to the sub-matrix that remains. Continue in this way until the entire matrix is in row - echelon form .

Step 6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

Example 3.2.12: Solve the following system of linear equations by using the Gauss - Jordan elimination method:

$$5x_1 + 6x_2 = 7$$
$$3x_1 + 4x_2 = 5$$

Solution: The system of linear equations has the following augmented matrix:

$$\left(\begin{array}{cc|c} 5 & 6 & 7 \\ 3 & 4 & 5 \end{array}\right) \xrightarrow{\frac{1}{5}R_1 \to R_1}$$

$$\left(\begin{array}{cc|c} 1 & \frac{6}{5} & \frac{7}{5} \\ 3 & 4 & 5 \end{array}\right) \xrightarrow{R_2 - 3R_1 \to R_2}$$

$$\left(\begin{array}{cc|c} 1 & \frac{6}{5} & \frac{7}{5} \\ 0 & \frac{2}{5} & \frac{4}{5} \end{array}\right) \xrightarrow{\frac{5}{2}R_2 \to R_2}$$

$$\left(\begin{array}{cc|c} 1 & \frac{6}{5} & \frac{7}{5} \\ 0 & 1 & 2 \end{array}\right) \xrightarrow{R_1 - \frac{6}{5}R_2 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right)$$

The last matrix is in reduced row - echelon form . The corresponding reduced system is:

$$\begin{aligned} x_1 \quad &= -1 \\ x_2 &= 2 \end{aligned}$$

Therefore the solution of the system is $x_1 = -1$, and $x_2 = 2$.

<u>Example 3.2.13:</u> Solve the following linear system using the Gauss - Jordan elimination method:

$$\begin{aligned} 4y + 2z &= 1 \\ 2x + 3y + 5z &= 0 \\ 3x + y + z &= 11 \end{aligned}$$

<u>Solution:</u> The system of linear equations has the following augmented matrix:

$$\left(\begin{array}{ccc|c} 0 & 4 & 2 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 1 & 1 & 11 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ 0 & 4 & 2 & 1 \\ 3 & 1 & 1 & 11 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 4 & 2 & 1 \\ 3 & 1 & 1 & 11 \end{array}\right) \xrightarrow{R_3 - 3R_1 \to R_3}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 4 & 2 & 1 \\ 0 & -\frac{7}{2} & -\frac{13}{2} & 11 \end{array}\right) \xrightarrow{\frac{1}{4}R_2 \to R_2}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & -\frac{7}{2} & -\frac{13}{2} & 11 \end{array}\right) \xrightarrow{R_3 + \frac{7}{2}R_2 \to R_3}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & -\frac{19}{4} & \frac{95}{8} \end{array}\right) \xrightarrow{-\frac{4}{19}R_3 \to R_3}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{5}{2} \end{array}\right) \xrightarrow{R_2 - \frac{1}{2}R_3 \to R_2}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{5}{2} \end{array}\right) \xrightarrow{R_1 - \frac{5}{2}R_3 \to R_1}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{2} & 0 & \frac{25}{4} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{5}{2} \end{array}\right) \xrightarrow{R_1 - \frac{3}{2}R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{5}{2} \end{array}\right)$$

The last matrix is in reduced row - echelon form . The corresponding reduced system is:

$$\begin{aligned} x \quad\quad &= 4 \\ y \quad &= \frac{3}{2} \\ z &= -\frac{5}{2} \end{aligned}$$

Therefore the solution of the system is $x = 4$ **,** $y = \frac{3}{2}$ **, and** $z = -\frac{5}{2}$ **.**

Example 3.2.14: Solve the following linear system using the Gauss - Jordan elimination method:

$$\begin{aligned} 3x_1 + 6x_2 - 9x_3 &= 15 \\ 2x_1 + 4x_2 - 6x_3 &= 10 \\ -2x_1 - 3x_2 + 4x_3 &= -6 \end{aligned}$$

Solution: The system of linear equations has the following augmented matrix:

$$\left(\begin{array}{ccc|c} 3 & 6 & -9 & 15 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right) \xrightarrow{\frac{1}{3}R_1 \to R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right) \xrightarrow{R_2 - 2R_1 \to R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ -2 & -3 & 4 & -6 \end{array}\right) \xrightarrow{R_3 + 2R_1 \to R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 - 2R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The last matrix is in reduced row - echelon form. The corresponding reduced system is:

$$\begin{aligned} x_1 + \quad\quad x_3 &= -3 \\ x_2 - 2x_3 &= 4 \end{aligned}$$

Thus $\quad x_1 = -x_3 - 3$

$\quad\quad\quad x_2 = 2x_3 + 4$

If we let $x_3 = t$**, then for any real number** t**,**

$$\begin{aligned} x_1 &= -t - 3 \\ x_2 &= 2t + 4 \\ x_3 &= t \end{aligned}$$

The system has infinite number of solutions .

If $t = 0$**, the solution will be** $x_1 = -3$, $x_2 = 4$ **and** $x_3 = 0$

If $t = -2$**, the solution will be** $x_1 = -1$, $x_2 = 0$ **and** $x_3 = -2$

If $t = 3.5$**, the solution will be** $x_1 = -6.5$, $x_2 = 11$ **and** $x_3 = 3.5$.

Exercises 3.2.15: **Solve each of the following linear systems using the Gauss - Jordan elimination method:**

(1)
$$\begin{aligned} 3x_1 + 6x_2 - 9x_3 &= 15 \\ 2x_1 + 4x_2 - 6x_3 &= 10 \\ -2x_1 - 3x_2 + 4x_3 &= -6 \end{aligned}$$

(2) $$\begin{array}{rcrcrcr} x_1 &+& x_2 &+& x_3 &=& 2 \\ 2x_1 &+& 3x_2 &-& x_3 &=& 9 \\ x_1 &+& 3x_2 &+& 2x_3 &=& 5 \end{array}$$

Remark 2.3.16: If we use Gauss or Gauss - Jordan eliminations methods and at any point in the process we obtain a row with all zeros to the left of the vertical line and a nonzero number to the right, we should stop, since we will have a contradiction: 0 = n and n ≠ 0. We can then conclude that the system has no solution.

Example 3.2.17: Solve the following linear system using the Gauss or Gauss - Jordan elimination method:

$$\begin{array}{rcrcrcr} 2x &-& 4y &+& z &=& -4 \\ 4x &-& 8y &+& 7z &=& 2 \\ -2x &+& 4y &-& 3z &=& 5 \end{array}$$

Solution: The system of linear equations has the following augmented matrix

$$\left(\begin{array}{rrr|r} 2 & -4 & 1 & -4 \\ 4 & -8 & 7 & 2 \\ -2 & 4 & -3 & 5 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1}$$

$$\left(\begin{array}{rrr|r} 1 & -2 & \frac{1}{2} & -2 \\ 4 & -8 & 7 & 2 \\ -2 & 4 & -3 & 5 \end{array}\right) \xrightarrow{R_2 - 4R_1 \to R_2}$$

$$\left(\begin{array}{rrr|r} 1 & -2 & \frac{1}{2} & -2 \\ 0 & 0 & 5 & 10 \\ -2 & 4 & -3 & 5 \end{array}\right) \xrightarrow{R_3 + 2R_1 \to R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & -2 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & -2 & 1 \end{array}\right) \xrightarrow{\frac{1}{5}R_2 \to R_2}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{array}\right) \xrightarrow{R_3 + 2R_2 \to R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{array}\right)$$

We stop the process of the Gauss or the Gauss - Jordan elimination , since the last row produces a contradiction .

Therefore the system has no solution .