# Chapter 3:
# Solving System of Linear Equations

## S1: Definition of a Matrix, and Types of Matrices

**Definitions 3.1.1:** A **matrix** is a rectangular array of numbers enclosed in parentheses. The numbers occurring in a matrix are called the **entries**.

Each matrix has a certain number of rows and a certain number of columns.

A matrix A with m rows and n column is called an **m × n matrix**. The numbers m and n are called the dimensions of the matrix A and $m \times n$ is called the **size** of the matrix A .

**Examples 3. 1.2:** i- $\begin{pmatrix} 3 & 1 \\ 7 & 6 \end{pmatrix}$ is a $2 \times 2$ matrix .

ii- $\begin{pmatrix} 5 & 4 & 9 \\ 9 & 2 & 0 \end{pmatrix}$ is a $2 \times 3$ matrix .

iii- $\begin{pmatrix} 2 & 7 & 9 \\ 1 & 0 & 7 \\ 8 & 4 & 3 \end{pmatrix}$ is a $3 \times 3$ matrix .

**Definition 3.1.3:** The entry whose position in the $i$ th row and the $j$ th column of a matrix A is called the $ij$**-entry** of the matrix A and denoted by $a_{ij}$ .

**Example 3.1.4:** The entries of the matrix $A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 7 & 6 \end{pmatrix}$ are $a_{11} = 3$, $a_{12} = 2$, $a_{13} = 4$ , $a_{21} = 1$ , $a_{22} = 7$ , $a_{23} = 6$ .

**Definition 3.1.5:** An $m \times m$ matrix is called a **square matrix of order m** .

**Examples 3.1.6:** i- The matrix $A = \begin{pmatrix} 1.2 & 3 & 2.6 \\ 7 & 0.3 & 9 \\ 2.5 & 0 & 4.1 \end{pmatrix}$ is a square matrix of order 3.

ii- The matrix $B = \begin{pmatrix} 3 & 4 \\ 1.2 & 6 \end{pmatrix}$ is a square matrix of order 2.

**Definitions 3.1.7:** A matrix with only one row is called a **row matrix**. A matrix with only one column is called a **column matrix**.

**Examples 3.1.8:** i- The matrix $D = \begin{pmatrix} 7 & 2 & 3 & 9 \end{pmatrix}$ is a row matrix .

ii- The matrix $E = \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}$ is a column matrix .

**Definition 3.1.9:** A square matrix A whose entries $a_{ij} = 0$ if $i \neq j$ is called a **diagonal matrix** .

**Examples 3.1.10:** i- $A = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ is a diagonal matrix .

ii- $B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3.2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 13 \end{pmatrix}$ is a diagonal matrix .

**Definition 3.1.11:** A diagonal matrix whose entries $a_{ii}$ are all equal some fixed number c is called a **scalar matrix** .

**Examples 3.1.12:** i- $A = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$ is a scalar matrix.

ii- $B = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$ is a scalar matrix.

**Definitions 3.1.13:**

A matrix whose entries are all zeros is called a **zero matrix** and denoted by **O**.

A diagonal matrix whose entries on the diagonal are all ones is called an **identity matrix** and denoted by **I**.

**Examples 3.1.14:**

1- $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a zero matrix.

2- $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is a zero matrix.

3- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an identity matrix.

4- $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an identity matrix.

## S2: Solving System of Linear Equations Using Gauss and Gauss - Jordan Eliminations Methods

**Definition 3.2.1:** A matrix B is said to be in **row - echelon form** if B satisfies the following properties :

1. If a row does not consist entirely of zeros , then the first nonzero number in the row is a 1 . ( We call this a **leading 1** ) .
2. If there are any rows that consist entirely of zeros , then they are grouped together at the bottom of the matrix .
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row .

**Examples 3.2.2:** The following matrices are in row-echelon form :

1) $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

2) $B = \begin{pmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3) $C = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{pmatrix}$

4) $D = \begin{pmatrix} 1 & 9 & 5 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

**Definition 3.2.3:** A matrix B in row - echelon form is said to be in **reduced row - echelon form** if B satisfies the following property :
Each column that contains a leading 1 has zeros everywhere else .

**Examples 3.2.4:** The following matrices are in reduced row - echelon form :

1) $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 9 \end{pmatrix}$

2) $B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

**Definition 3.2.5:** The **augmented matrix** for a system of m linear equations in n unknowns is the $m \times (n+1)$ matrix of the coefficients of the unknowns and the constants , and the coefficients of the unknowns are separated from the constants by a vertical line .

**Example 3.2.6:** The following system of linear equations

$$\begin{aligned} 9x_1 + 4x_2 &= 5 \\ 7x_1 + 2x_2 &= 6 \end{aligned}$$

has the following augmented matrix

$$\left(\begin{array}{cc|c} 9 & 4 & 5 \\ 7 & 2 & 6 \end{array}\right)$$

**Example 3.2.7:** The following system of linear equations

$$\begin{aligned} 2x + 4y - 3z &= 1 \\ x + y + 2z &= 9 \\ x + 2y - z &= 2 \end{aligned}$$

has the following augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 4 & -3 & 1 \\ 1 & 1 & 2 & 9 \\ 1 & 2 & -1 & 2 \end{array}\right)$$