Chapter one
The Real numbers

Introduction :-

1. The set of Natural numbers: $N = \{0, 1, 2, 3, 4, ----\}$
   or $N = \{1, 2, 3, ----\}$

2. The set of integer numbers $Z = \{----, -3, -2, -1, 0, 1, 2, 3, ---\}$

3. The set of rational numbers.
   Def :- A number of the Form $\frac{m}{n}$ where m and n are integers, $n \neq 0$ is called rational number.
   i.e $Q = \{ \frac{m}{n},\ m, n \in Z,\ n \neq 0 \}$.

4. The set of irrational numbers.
   Def. A number where is not rational number is called irrational number.

Ex :- show that $\sqrt{2}$ is not rational number.

sol, suppose that $\sqrt{2}$ is rational number.

That means $\sqrt{2} = \frac{m}{n} \Rightarrow 2 = \frac{m^2}{n^2} \Rightarrow m^2 = 2n^2$

1. if m is even and n is odd.
   $\therefore m = 2K \Rightarrow m^2 = 4K^2 \Rightarrow 4K^2 = 2n^2 \Rightarrow 2K^2 = n^2$
   $\therefore n$ is even (contradiction)

2

2. if m is odd and n are even.
we get $n = 2K \Rightarrow n^2 = 4K^2$
since $m^2 = 2n^2 \rightarrow m^2 = 2(4K^2)$. $\Rightarrow m^2 = 8K^2$
we get m^2 is even (contradiction).
but m is odd then m^2 is odd

3. if m and n are odd
m is odd $\Rightarrow m^2$ is odd.
but $m^2 = 2n^2$ and $2n^2$ is even.
$\therefore m^2$ is even (contracadiction).

Then $\sqrt{2}$ is not rational number.

5. The set of Real numbers $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.
$\mathbb{R} = \{x : x \text{ is real numbers}\}$

6. The set of complex number $\mathbb{C} = \{x+iy : x, y \in \mathbb{R}, i = \sqrt{-1}\}$

Intervals :-

1. (a,b) open intervals 

   $(a,b) = \{ a < x < b : x \in \mathbb{R} \}$.

2. $[a,b]$ closed intervals 

   $[a,b] = \{ a \le x \le b : x \in \mathbb{R}$

3. $[a,b)$ half closed intervals From left 

   $[a,b) = \{ a \le x < b : x \in \mathbb{R} \}$.

4. $(a,b]$ half closed intervals From right 

   $(a,b] = \{ a < x \le b : x \in \mathbb{R} \}$.

Order relation

Definition :- Let S be a set, an order on S is a relation denoted by $<$, with the Following two properties:

(i) If $x \in S$ and $y \in S$ then one and only one of The statments is true.

$$x < y , \quad x = y , \quad x > y .$$

(ii) if $x, y, z \in S$, if $x < y$ and $y < z$ then $x < z$.

Definition : An order set is a set S in which an order is defined.

The pair $(S, <)$ is called order set.

Ex③ let Q be the set of rationall number, defined a relation ($<$) on Q as follow $a<b$ mean $b-a$ is positive rational number for $a,b \in Q$.

Solution: ① if $a,b \in Q$ then $a<b$.

② if $a,b,c \in Q$ and $a<b$, $b<c$

To prove $a<c$

since $a<b$ we get $b-a$.

$b<c$ we get $c-b$

Now we have $b-a$ ....①

$c-b$ ....② بالجمع

$$\frac{\begin{array}{c} b-a \\ c-b \end{array}}{c-a} \Rightarrow a<c$$

Definition :- Suppose that S is an ordered set, and $E \subseteq S$. If there exists $b \in S$ such that i. $x \le b$ for every $x \in E$, we say that E is bounded above, and call b is an upper bound of E.

ii $x \ge b$ for every $x \in E$, we say that E is Lower bounded and call b is an Lower bound of E.

i.e ① S is order set and $E \subseteq S$ then E is bounded above if $\exists\, b \in S$ s.t $x \le b \ \ \forall x \in E$

(ii) S is order set and $E \subseteq S$, then E is bounded below if $\exists\, b \in S'$ s.t $b \le x \ \ \forall x \in E$.

Ex :- ① Let $S = Q$, $E = \{2, 3, 4, 5, 6, 7\}$.

Then The upper bound are $7, 8, 9, 10, ----$

and the Lower bound are $2, -1, 0, -1, -2, ---$

$\therefore$ E is bounded above and bounded below.

② Let $S = Z$, $E = \{----, -3, -2, -1, 0, 1,\}$

Then the upper bound are $1, 2, 3, 4 ----$

since $1 \in Z$ and $1 \ge x \ \ \forall x \in E$

and E is bounded above.

but E is not bounded below since $\nexists\, b \in S$ s.t $b \le x \ \ \forall x \in E$

Definition :- An order Set S is Said to be greatest Lower bound property, if $E \subseteq S$, $E \neq \phi$. and E is bounded below, inf (g-L-b-)(E) exist in S.

Ex, $S = \mathbb{R}$, $E = \{x : 0 \leq x \leq 1\}$.

$\mathbb{R}$ is greatest - Lower bound proprty

Since 1- $E \subseteq \mathbb{R}$ 2- $E \neq \phi$, 3- E is bounded below.

4. $\inf(E) = g.l.b(E) = 0$ exist in $\mathbb{R}$.

Definition := An order set S is said to be least upper bound property if $E \subseteq S$, $E \neq \phi$ and E bounded above, sup (L.u.b)(E) exist in S.

Ex ① Let $S = \mathbb{R}$, $E = \{x : 0 < x < 1\}$.

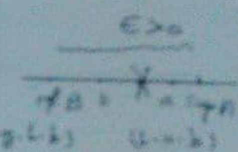
$S = \mathbb{R}$ is least upper bound property

Since
1. $E \subseteq \mathbb{R}$.
2. $E \neq \phi$.
3. E is bounded above.
4. $\sup(E) = \text{L.u.b}(E) = 1$. exist in $\mathbb{R}$.

Examples:-

Let $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$ s.t $\forall a \in A, b \in B$. $a < b$ then $\sup(A) \le \inf(B)$

Solution:- suppose that $\sup(A) > \inf(B)$.



$\therefore \inf(B) + \frac{\epsilon}{2} = \sup(A) - \frac{\epsilon}{2}$

$\exists\, a \in A$ s.t $\sup(A) - \frac{\epsilon}{2} < a < \sup(A)$

$\exists\, b \in B$ s.t $\inf(B) < b < \inf(B) + \frac{\epsilon}{2}$.

Since $\inf(B) + \frac{\epsilon}{2} = \sup(A) - \frac{\epsilon}{2}$.

$\therefore b < \sup(A) - \frac{\epsilon}{2} < a$.

$\therefore b < a$ C!

$\therefore \sup(A) \le \inf(B)$.

Difinition :-

An order Set S is called well-order if every non-empty subset of S has a smallest (or first) element.

Examples. ① $S = N$, $(N, <)$ is an order set

let $E = \{7, 8, 9, 10\}$.

$G = \{2, 3, 4\}$.

since $G \subseteq N$ has smallest element.

$E \subseteq N$ has smallest element.

$\therefore N$ is well-order.

② $S = Z$, $(Z, <)$ an order Set.

let $E = \{ ----, -3, -2, -1, 0\}$

$E \subseteq Z$, but E has no smallest element.

$\therefore Z$ is not well-order.

FiLD

Definition :-

A Field is a set F with two operation called addition $(+)$ and multiplication $(\cdot)$ which satisfy the Following.

a $(F,+)$ is Commutitive group.

i.e 1. if $x, y \in F$ then $x+y \in F$.

2. if $x, y \in F$ then $x+y = y+x$.

3. if $x, y, z \in F$ then $(x+y)+z = x+(y+z)$.

4. $\exists$ an element $0 \in F$ s.t $x+0 = 0+x = x \quad \forall x \in F$.

5. For every element $x \in F$, $\exists$ an element $-x \in F$ s.t $x+(-x) = -x+x = 0$

(1-5) is called axiom For addition.

b. $(F-\{0\}, \cdot)$ is Commutitive group.

i.e 1. if $x, y \in F$ then $x \cdot y \in F$.

2. if $x, y \in F$ then $x \cdot y = y \cdot x$

3. if $x, y, z \in F$ then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

4. $\exists$ an element $1 \in F$ s.t $x \cdot 1 = 1 \cdot x = x \quad \forall x \in F$

5. For every element $x \in F$, then $\exists\ x^{-1} \in F$ s.t $x \cdot x^{-1} = x^{-1} \cdot x = 1$

(1-5) called axiom For Multiplication.

c. The distributive Law : if $x, y, z \in F$ then
$x \cdot (y+z) = x \cdot y + x \cdot z$, $(x+y)z = xz+yz$
$\forall x, y, z \in F$.

Examples:

1. $(\mathbb{R}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$ are Fields
2. $(\mathbb{C}, +, \cdot)$ is Field
3. $(\mathbb{Z}_2, +_2, \cdot_2)$ is Field

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

Example: $(\mathbb{Q},+,\cdot,<)$ is an order Field.

Sol:- 1. Let $x,y\in\mathbb{Q}$. (T.P. $(\mathbb{Q},+)$ is Commutitive group).

$\Rightarrow x=\frac{a}{b},\ y=\frac{c}{d}\quad \forall a,b,c,d\in\mathbb{Z},\ b,d\neq 0.$

$\therefore\ x+y=\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}\in\mathbb{Q}.$

2. if $x+y\in\mathbb{Q}$.

$\Rightarrow x+y=\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}=\frac{bc+ad}{bd}=y+x.$

$\therefore\ x+y=y+x.$

3. if $x,y,z\in\mathbb{Q}$

$$x+(y+z)=\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)=\frac{a}{b}+\frac{fc+de}{df}=$$

$$\frac{adf+bcf+bde}{bdf}\ \text{---}\ (1)$$

$$(x+y)+z=\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\left(\frac{ad+bc}{bd}\right)+\frac{e}{f}$$

$$=\frac{adf+bcf+bde}{bdf}\ \text{---}\ (2)$$

$\therefore\ (x+y)+z=x+(y+z).$

4. $\exists\, 0\in\mathbb{Q}$ s.t $0+\frac{a}{b}=\frac{a}{b}+0=\frac{a}{b}$

$\therefore\ x+0=0+x=x\quad\forall x\in\mathbb{Q}.$

5. For every $x\in\mathbb{Q}$, $\exists -x\in\mathbb{Q}$ s.t $x+(-x)=-x+x=0$

$\frac{a}{b}+\left(-\frac{a}{b}\right)=-\frac{a}{b}+\frac{a}{b}=0.$

(17)

b. T.P $(Q, \cdot)$ is Commutitive group.

1. if $x, y \in Q$, $x = \frac{a}{b}$, $y = \frac{c}{d}$.

then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \in Q$.

2. if $x, y \in Q$.

$x \cdot y = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} = \frac{ca}{db} = \frac{c}{d} \cdot \frac{a}{b} = y \cdot x.$

$\therefore$ $x \cdot y = y \cdot x.$

3. if $x, y, z \in Q$.

$x \cdot (y \cdot z) = \frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) = \frac{ace}{bdf}$ --- ①.

$(x \cdot y) \cdot z = \left(\frac{a}{b} \cdot \frac{c}{d}\right) \frac{e}{f} = \frac{ace}{bdf}$ --- ②

$\therefore$ $x \cdot (y \cdot z) = (x \cdot y) \cdot z.$

Definition :- $(F, +, \cdot, <)$ is called order Field if

1. $(F, +, \cdot)$ is a Field.
2. $(F, <)$ is an order Set.
3. IF $a \leq b$ then $a + c \leq b + c$ $\forall a, b, c \in F$
4. IF $a, b \in F$. $a > 0$, $b > 0$ then $a \cdot b > 0$.