Definition

Let $(F,+,\cdot,<)$ is an order field, then $(F,+,\cdot,<)$ is called complete order field if every subset $E \subseteq F$ which is bounded above and has Least upper bound (L.u.b) in F.

Example:-

The real number is complete order Field.

Completnees - property of $\mathbb{R}$:

Every non-empty set of real number which is bounded above has least upper bound in $\mathbb{R}$.

proposition :-

Every order Field contians the integer number.

proof :-

Let $(F, +, \cdot, <)$ be an order Field and 0 and 1 are the addictive identity and multiplicative identity of F respectively.

Now, $1+1 \in F$, we claim that $1+1 \neq 1$ and $1+1 \neq 0$

suppose $1+1 = 1$

$$\Rightarrow -1 + (1+1) = -1+1$$

$$(-1+1) + 1 = -1+1$$

$$0 + 1 = 0$$

$$1 = 0 \quad C! \quad \text{since } F \text{ is an order Field.}$$

$\therefore 0 < 1$, $0+0 < 1+1$.

$$0 < 1+1$$

But $1+1 = 2$ and $2 \neq 1$

simiarly we can prove $3 = 1+1+1 \neq 2$.

By induction $n = 1+1+\cdots+1 \in F$

since F is a Field and $n \in F \Rightarrow -n \in F$.

$\therefore F$ contrians the integer number.

Corollary: Every order field contains the field of rational numbers (Exc)

Example Show that the equation $x^2=2$ has no roots in The field rational numbers.

proof Suppose that y is rational number $\ni y^2=2$

let $y=\frac{m}{n}$, m,n are positive integer and $n\neq 0$

g.c.d$(m,n)=1$.

$$\left(\frac{m}{n}\right)^2=2 \longrightarrow \frac{m^2}{n^2}=2 \longrightarrow m^2=2n^2$$

1- Suppose m is even and n is odd

$m=2k \longrightarrow m^2=4k^2$

$4k^2=2n^2 \longrightarrow 2k^2=n^2 \Rightarrow$ we get n^2 is even

but (n) is odd $\therefore n^2$ is even and odd C!

(which is contradiction)

2- suppose m is odd and (n) is even

$n=2k \longrightarrow n^2=4k^2$

$\therefore m^2=2\cdot 4k^2 \longrightarrow m^2$ is even but m is odd $\rightarrow m^2$ is odd

$\therefore m^2$ is even and odd which is contradiction (C!)

3- suppose m and n are odd

$m^2=2n^2 \Rightarrow m^2$ is even, But m is odd $\rightarrow m^2$ is odd

$\therefore m^2$ is even and odd which is contradiction

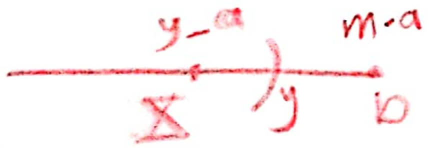
There is no rational number which is satisfy the equation $x^2=2$

($\nexists$ a solution of the equation $x^2=2$ in Q)

Archimedean property

If $a, b \in \mathbb{R}$, and $a > 0$ then there exist positive integer n such that $n \cdot a > b$

Proof Let $X = \{ka : k \in N\}$



$\therefore \phi \neq X \subseteq \mathbb{R}$

Suppose that The Theorem is not true

i.e. $\forall n \in N$, $n \cdot a < b \rightarrow b$ is upper bound of X

Since $X \neq \phi$, $X \subseteq \mathbb{R}$ since $\mathbb{R}$ is complete order field

$\Rightarrow X$ is bounded above

$\Rightarrow \exists$ Least upper bound of X say y

i.e. $y = \sup(X)$

$\because a > 0 \Rightarrow -a < 0 \Rightarrow y - a < y$

$\therefore y - a$ is not upper bound of X

$\therefore \exists\ \underbrace{m \cdot a}_{\text{upper bound}} \in X$ s.t $y - a < m \cdot a$

$\Rightarrow ma + a > y$

$\Rightarrow a(m+1) > y$

$\Rightarrow a(m+1) \in X$ C!

Since There is an element $a(m+1)$ in $X \ni a(m+1) > b$

Since y is Least upper bound $= L.u.b$ of X

$\therefore n \cdot a > b$ is true

Corollary Every real number $\epsilon > 0$, There exist positive integer n such that $\frac{1}{n} < \epsilon$. (Exc)