The density of The rational numbers:

If $a,b \in \mathbb{R}$ s.t. $a<b$ then $\exists r \in \mathbb{Q}$ s.t $a<r<b$

The density of irrational number

Theorem: If $a,b \in \mathbb{R}$ s.t. $a<b$ then there is K irrational number s.t. $a<K<b$.

Proof: suppose that the theorem is not true.

$\therefore \forall x \in \mathbb{R}$ and $a<K<b$, K is rational number

$\therefore a+\sqrt{2} < K+\sqrt{2} < b+\sqrt{2}$

$\because K+\sqrt{2}$ is irrational number Then there is no rational number between $a+\sqrt{2}$ and $b+\sqrt{2}$ which is contradaction with density of rational numbers

$\therefore a<K<b$ and K is irrational number

Corollary: If $a,b \in \mathbb{R}$ and $a<b$, Then the set of irrational numbers between a and b is infinite

Proof (Exc)

Definition: A complex number is an order pair where $a,b \in \mathbb{R}$ and we denoted by the set of all complex number by $\mathbb{C}$

Definition: Let (a,b), (c,d) be two complex numbers.

Then:

① $(a,b)=(c,d)$ iff. $a=c$, $b=d$

② $(a,b)+(c,d)=(a+c\ ,\ b+d)$

③ $(a,b)\cdot(c,d)=(ac-bd\ ,\ ad+bc)$

Theorem The Trip $(\mathbb{C},+,\cdot)$ is a field

Proof: Let $x=(a,b)$, $y=(c,d)$

If $x,y\in\mathbb{C}$ Then

① $x+y=(a,b)+(c,d)=(a+c\ ,\ b+d)\in\mathbb{C}$

$\therefore\ x+y\in\mathbb{C}$

② $x+y=(a,b)+(c,d)=(a+c,\ b+d)=(c+a\ ,\ d+b)$
$=y+x$

$\therefore\ x+y=y+x$

③ If $x,y,z\in\mathbb{C}$ Then $(x+y)+z=((a,b)+(c,d))+(e,f)$

$$=(a+c,\ b+d)+(e,f)$$
$$=(a+c+e\ ,\ b+d+f)$$
$$=(a,b)+(c+e,\ d+f)=x+(y+z)$$

$\therefore\ (x+y)+z=x+(y+z)$

④ $\exists$ an element $0\in\mathbb{C}$ s.t. $x+0=0+x=x$

we take $0=(0,0)$

$\Rightarrow x+0=(a,b)+(0,0)=(a+0\ ,\ b+0)=(a,b)=x$

(5) For every element $x \in \mathbb{C}$, $\exists -x \in \mathbb{C}$ s.t.

$x+(-x)=0$

we take $-x=(-a,-b)$

$\therefore x+(-x)=(a,b)+(-a,-b)=((a+c-a),(b+c-b))$
$=(0,0)$

(6) If $x,y \in \mathbb{C}$

$\therefore x\cdot y=(a,b)\cdot(c,d)=(ac-bd,ad+bc) \in \mathbb{C}$

$\therefore x\cdot y \in \mathbb{C}$

(7) If $x,y \in \mathbb{C}$

$x\cdot y=(a,b)\cdot(c,d)=(ac-bd, ad+bc)$
$=(ca-db, da+cb)=y\cdot x$

$\therefore x\cdot y=y\cdot x$

(8) $x,y,z \in \mathbb{C}$

$x\cdot(y\cdot z)=(a,b)((c,d)\cdot(e,f))$
$=(a,b)(ce-df, cf+de)$

$=((ace-adf)-(bcf+bde),(acf+ade)+(bce-bdf))$

$=(ac-bd,ad+bc)\cdot(a,b)=(x\cdot y)\cdot z$

(9) $\exists$ an element $1 \in \mathbb{C}$ s.t. $x\cdot 1=x$

we take $1=(1,0)$

$\Rightarrow (a,b)\cdot(1,0)=(a,b)=x$

(10) For every element $x \in \mathbb{C}$ $\exists$ $x^{-1}=\frac{1}{x} \in \mathbb{C}$

$\ni x^{-1}\cdot x=1$

we Take $x^{-1}=\frac{1}{x}=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)$

$\therefore (a,b)\cdot\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)=(1,0)=1$

(11) The distributive Law. (Exc)

$x\cdot(y+z) = x\cdot y + x\cdot z$

$\therefore (\mathbb{C}, +, \cdot)$ is field

Remarks:

(1)- The element i is defined as $i = (0,1)$

(2)- $i^2 = -1$

Proof $i^2 = i\cdot i = (0,1)\cdot(0,1) = (0\cdot 0 - 1\cdot 1, 0\cdot 1 + 0\cdot 1)$
$= (-1, 0) = -1$

(3)- $(a,b) = a + ib$

Proof $a+ib = (a,0) + (0,1)\cdot(b,0)$
$= (a,0) + (b\cdot 0 - 1\cdot 0, 0\cdot 0 + 1\cdot b)$
$= (a,0) + (0,b)$
$= (a+0, b+0) = (a,b)$

(4)- The conjugate of complex number

Let $z = a+ib$ is a complex number then $\bar{z} = a - ib$ is called the conjugate of z s.t. the number a is called the real part and b is called the imaginary part.