Countable sets

Remarks:-

1. A Function $f: A \to B$ is said to be one-two-one (1.1) if $\forall\, a, b \in A$ then $f(a) = f(b)$ iff $a = b$.

2. A Function $f: A \to B$ is said to be onto if $f(A) = B$

3. if f is 1.1 and onto then f is 1.1 correspondence. and written $A \sim B$.

4. $J = \{1, 2, \dots\}$
   $J_n = \{1, 2, 3, \dots, n\}$

Definition:-

A set X is said to be finite if its empty or equivalent the set J_n for some positive integer.
A set which is not finite is called infinite.

Example: let $A = \{14, 12, 30, 15\}$, $J = \{1, 2, 3, 4\}$
s.t $f(12) = 2$, $f(30) = 1$, $f(14) = 4$, $f(15) = 3$
$\therefore$ f is 1-1 and onto (1-1 correspondence)
we get $A \sim J_4$
$\therefore$ A is finite.

Definition :-

A set A is said to be countable if there exist 1-1 and onto function f from A onto J. ($A \sim J$).

proposition : Every finite set is countable.

Example:-

The set of all integers is countable.

proof :- Let $f: J \longrightarrow Z$ be a function defined by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{-x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

1- To show that f is 1-1.

Let $a, b \in J$ s-t $a \neq b$.

(a) if a, b are even

$\therefore f(a) = \frac{a}{2}$, $f(b) = \frac{b}{2}$ since $a \neq b \Rightarrow \frac{a}{2} \neq \frac{b}{2}$

$\therefore f(a) \neq f(b)$.

(b) if a, b are odd.

$f(a) = \frac{-a+1}{2}$, $f(b) = \frac{-b+1}{2}$ since $a \neq b \Rightarrow \frac{-a+1}{2} \neq \frac{-b+1}{2}$

$\therefore f(a) \neq f(b)$.

(c) if a is even, b is odd (Exc) (d) if a is odd, b is even (Exc)

Thus f is 1-1

2. To show f is onto

$\forall b \in \mathbb{Z}$ T.P. $\exists a \in J$ s.t $f(a) = b$.

(a) if a is even $\Rightarrow f(a) = \frac{a}{2} \Rightarrow \frac{a}{2} = b \Rightarrow a = 2b$.

$\therefore f(a) = \frac{a}{2} = \frac{2b}{2} = b$

$\therefore f(a) = b$.

(b) if a is odd. then $f(a) = \frac{-a+1}{2} \Rightarrow \frac{-a+1}{2} = b$

$\Rightarrow a = -2b+1$

$\therefore f(a) = \frac{-a+1}{2} = \frac{-(-2b+1)+1}{2} = b$.

$\therefore f(a) = 2$.

$\therefore f$ is onto.

$\therefore \mathbb{Z} \sim J$.

$\therefore \mathbb{Z}$ is countable.

<u>Definition</u>:- A set A is called at most countable if A either finite or countable.

<u>Definition</u>:- A set A which is not countable and not finite is called uncountable.

<u>Example</u>:- The set of real numbers is uncountable.

<u>proof</u>:- Let $S \subseteq \mathbb{R}$ and S is countable, we must to prove $S \neq \mathbb{R}$.

Let $S = \{a_1, a_2, \ldots, a_n, \ldots\}$ and.

Let I_1 be interval s.t $|I_1| < 1$ and $a_1 \notin I_1$

Let I_2 be interval s.t $|I_2| < \frac{1}{2}$ and $a_2 \notin I_2$, $I_2 \subseteq I_1$

" I_3 " " " $|I_3| < \frac{1}{3}$ and $a_3 \notin I_3$, $I_3 \subseteq I_2$.

⋮

Let I_n be interval s.t $|I_n| < \frac{1}{n}$ and $a_n \notin I_n$, $I_n \subseteq I_{n-1}$

we have

$$\bigcap_{n=1}^{\infty} I_n = \{x\} \Rightarrow x \in I_n \quad \forall n.$$

since $a_n \notin I_n$ $\therefore a_n \neq x$.

$\therefore x \notin S \Rightarrow \mathbb{R} \neq S$.

$\therefore \mathbb{R}$ is not countable.

Definition:-

Let X be a set. A Function $f : J \to X$ is called sequence in X.

Remarks:-

1. For each $n \in J$ we denoted by the value $f(n)$ by $\{a_n\}$ or $\langle a_n \rangle$.
2. if $X = \mathbb{R}$, Then the sequence is called the sequence of real numbers.
3. if X is countable then the range of the sequence is X.

Theorem:- Every infinite subset of countable set is countable.

proof:- Let X be a countable set and A be an infinite subset of X.

$\Rightarrow X \sim J$

and $J \sim X$ (by def. of countable set)

Then X is a range of sequence.

element of X can be written $x_1, x_2, \ldots, \ldots$ which are distance element.

Let n_1 be the smallest positive integer $\ni x_{n_1} \in A$.

" n_2 " " " " " $\ni n_2 > n_1$ and $x_{n_2} \in A$

" n_3 " " " " " $\ni n_3 > n_2$ and $x_{n_3} \in A$.

$\vdots$ $\vdots$

n_k " " " " " $\ni n_k > n_{k-1}$ and $x_{n_k} \in A$.

Let $f: J \longrightarrow A$ defined by $f(k) = x_{n_k}$.

$\therefore f$ is 1-1 and onto.

$\therefore J \sim A \Rightarrow A \sim J$.

$\therefore A$ is countable set.

Theorem:- Let $\langle E_n \rangle$ be a sequence of countable set and let $S = \bigcup\limits_{n=1}^{\infty} E_n$, then S is countable.

proof:- Since E_n is countable for each n.

$\therefore J \sim E_n$ for each n.

elements of E_n can be arranged in a sequence of distinct elements of

$$E_1 = \{x_{11}, x_{12}, x_{13}, \dots\}$$
$$E_2 = \{x_{21}, x_{22}, x_{23}, \dots\}$$
$$E_3 = \{x_{31}, x_{32}, x_{33}, \dots\}$$
$$\vdots$$

$$S = E_1 \cup E_2 \cup E_3 \cup \dots$$

Arrange elements of S in a sequence as follow
$$S = \{x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, \dots\}.$$

if same element apear more than one we choose one of them only.

since E_1 is infinite and $E_1 \subseteq S$

$\therefore$ S is infinite

Define a function $f : J \longrightarrow S$ as follow.

$f(1) = x_{11}$, $f(2) = x_{21}$, $f(3) = x_{12}$, ...

Then f is 1-1 and onto.

$\therefore$ $J \sim S \Rightarrow S$ is countable.

<u>Corollary</u> :- The set of all rational number is countable.

proof : (Exc).

Theorem let A be a Countable set and $B_n = \{a_1, a_2, \dots, a_n\}, a_n \in A$ and $a_1, a_2, \dots, a_n$ need not be distenct then B_n is Countable.

proof :- (By Mathematical induction)

1. when $n = 1$

$B_1 = \{a_1 ; a_1 \in A\} = A$

since A is Countable then B_1 is Countable.

2. Suppose B_k is Countable

3. T.p B_{k+1} is Countable.

For any element $x \in B_k$, $B_x = \{(a,b), a \in B_x, b \in A\}$.

then $B_x \sim A$.

since A is Countable $\Rightarrow B_x$ is Countable and $\bigcup_{x \in B_x} B_x$ is Countable, But $\bigcup_{x \in B_x} B_x = B_x \times A$.

$\therefore B_x \times A$ is Countable.

$B_x \times A = B_{k+1} \Rightarrow B_{k+1}$ is Countable $\Rightarrow B_n$ is Countable

Corollary :- The set of irrational numbers is uncountable.

proof : (Exc).