Chapter two

Metric space الفضاء المتري

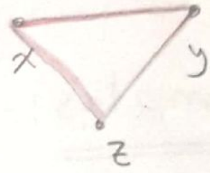
Definition :- let X be any set. A function $d: X \times X \to R$ is called Metric or distance function if it satisfy The following properties:-

(1) $d(x,y) > 0$ iff $x \neq y$

(2) $d(x,y) = 0$ iff $x = y$

(3) $d(x,y) = d(y,x)$ for any $x,y \in X$

(4) $d(x,y) \leq d(x,z) + d(z,y)$ for any $x,y,z \in X$



and the pair (X,d) is called metric space.

Remark :- $d(x,y)$ is called distance.

Examples :- let $X = R$, $d: X \times X \to R$ be a function defined by $d(x,y) = |x-y|$, then (X,d) is a metric space (is called usuall metric space).

Solution :-

① suppose $d(x,y) > 0$

$\Rightarrow |x-y| > 0 \Rightarrow x \neq y$

$\Leftarrow$ Conversely, suppose $x \neq y$

$\Rightarrow x-y \neq 0$

Thus $|x-y| > 0 \Rightarrow d(x,y) > 0$.

② suppose $d(x,y) = 0$

$\Rightarrow |x-y| = 0 \Rightarrow x-y = 0 \Rightarrow x = y$

$\Leftarrow$ Conversely, suppose $x = y$

$\Rightarrow x-y = 0 \Rightarrow x-y = 0 \Rightarrow |x-y| = 0$

$\Rightarrow d(x,y) = 0$

③ $d(x,y) = |x-y| = |y-x| = d(y,x)$

④ let $x,y,z \in R$

$$d(x,y) = |x-y| = |x-z+z-y| \leq |x-z| + |z-y|$$
$$\leq d(x,z) + d(z,y)$$

$\therefore$ d is ametric

(R,d) is ausual metric space.

Examples:- let X be any set

$d: X \times X \longrightarrow R$ defined by $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

Then d is ametric (is called trivial metric) and (X,d) is called metric space.

Solution:- ① suppose $d(x,y) > 0$

$\Rightarrow d(x,y) = 1 \Rightarrow x \neq y$

$\Leftarrow$ Converseley suppose $x \neq y$

$\therefore d(x,y) = 1 \Rightarrow d(x,y) > 0$

② suppose $d(x,y) = 0$

$\Rightarrow x = y$

$\Leftarrow$ converseley suppose $x = y$

$\Rightarrow d(x,y) = 0$

③ if $x \neq y \Rightarrow d(x,y) = 1$, $d(y,x) = 1$

$\therefore d(x,y) = d(y,x)$

if $x = y$

$\Rightarrow d(x,y) = 0$, $d(y,x) = 0$

$\therefore d(x,y) = d(y,x)$

④ Let $x,y,z \in X$ T.p $d(x,y) \leq d(x,z) + d(z,y)$

① if $x=y=z$

$d(x,y)=0$, $d(x,z)=0$, $d(z,y)=0$

$d(x,y) \leq d(x,z)+d(z,y)$

$0 \quad = 0+0=0$

② if $x \neq y \neq z$

$d(x,y)=1$, $d(x,z)=1$, $d(z,y)=1$

$d(x,y) \leq d(x,z)+d(z,y)$

$1 \quad \leq 1+1=2$

③ if $x=y\neq z$

$d(x,y)=0$, $d(x,z)=1$, $d(z,y)=1$

$d(x,y) \leq d(x,z)+d(z,y)$

$0 \leq 1+1=2$

$\therefore$ d is ametric, (x,d) is ametric space.

Example :- Let $X=R^2$, $d: R^2 \times R^2 \to R$ defined by

$$d(x_1,x_2)(y_1,y_2)=\sqrt{(y_2-x_2)^2+(y_1-x_1)^2}$$

Then d is ametric on R^2 (is called the usual metric on R^2

and (R^2,d) is ametric space.

Definition :- Let (X, d) be a metric space and let r be a positive number a neighbourhood of a point $p \in X$, the set of all points $x \in X \ni d(p, x) <$ and is denoted by $N_r(p)$

$$\therefore N_r(p) = \{x \in X : d(p, \overset{x}{r}) < r\}$$

Example :- let (R, d) be the usual metric space $r = 5$, $p = 2$

sol:

$$N_r(p) = \{x \in X : d(p, x) < r\}$$

$$N_5(2) = \{x \in R : d(2, x) < 5\}$$

$$= \{x \in R : |x - 2| < 5\}$$

$$= \{x \in R : -5 < x - 2 < 5\}$$

$$= \{x \in R : -3 < x < 7\}$$

Example :- let (R^2, d) beausual metric space.
Describle the niegbourhood of $(0,0)$ with radius 1.

Solution :-

$$N_1(0,0) = \{(x,y) \in \mathbb{R}^2 : d(0,0)(x,y) < 1\}$$
$$= \{(x,y) \in \mathbb{R}^2 : \sqrt{(y-0)^2 + (x-0)^2} < 1\}$$
$$= \{(x,y) \in \mathbb{R}^2 : \sqrt{y^2 + x^2} < 1\}$$
$$= \{(x,y) \in \mathbb{R}^2 : y^2 + x^2 < 1\}$$

set.

Definition:- let (X,d) be ametric space let $E \subseteq X$ apoint $p \in E$ is Called an interier point of E if $\exists N_r(p) \ni N_r(p) \subseteq E$. The set of all interier point of E denoted by E° or int E.

Example:- Let (R,d) be ausual metric space. $E = [0,1)$? Find E°.

Solution :- let $p \in [0,1)$

$$r = \frac{1}{2} \min\{|p-0|, |p-1|\}$$

$\therefore$ p is interier point of $[0,1)$

Let $p=0$

it is not possible to find $N_r(p) \ni$ $N_r(p) \subseteq E$

$\therefore$ int $E = (0,1)$

Example:- let (R,d) be ausual metric space, $E = \{4, 5, 6\}$? Find E°

Solution:- Since there is no niegbourhood of any point of E which is subset of E then $E^\circ = \phi$.

Definition:- Let (X,d) be ametric Space asubset E of X is Called openset if every point of E is an interier point of E.

Example:- ① $E = (0,1)$, $E = \{x : 0 < x < 1\}$

Since every point of $(0,1)$ is an interier point

$\therefore$ $(0,1)$ is open

② $A = [0,1]$, $A = \{x : 0 \le x \le 1\}$

Since (0) and (1) are in A and not interier point of A

$\therefore$ A is not open set.

Theorem:- Every nieghourhood is an open set.

Proof:- Let $N_r(p)$ be a niegh of p in metric space (X,d) and $x \in N_r(p)$ T.p x is interior of $N_r(p)$

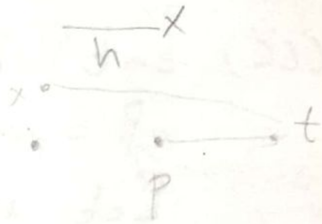
$\therefore d(x,p) < r$ (def of niegh)

$\Rightarrow r - d(x,p) > 0$

let $h = r - d(x,p)$

take $N_h(x)$ as niegh of X

T.P $N_h(x) \subseteq N_r(p)$



let $t \in N_h(x) \Rightarrow d(x,t) < h$

$\therefore d(p,t) \leq d(x,p) + d(x,t)$

$< r - \cancel{h} + \cancel{h} < r$

$\therefore d(p,t) < r \Rightarrow t \in N_r(p)$

$\therefore N_h(x) \subseteq N_r(p)$

$\therefore x$ is an interier point of $N_r(p)$

$\therefore N_r(p)$ is open set.

Theorem:- if (X,d) be a metric space. Then

(i) X, ϕ are open set

(ii) The intersection of any two open sets is ope

(iii) The Union of any family of open set is ope

Proof:- (i) if $x \in X$, $r > 0$, then $N_r(x) \subseteq X$

Then X is open.

The emply set is open becouse there are no point $x \in \phi$ and for each $x \in \emptyset$, $r >$

$N_r(x) \subseteq \phi$

$\therefore \phi$ is open set.

(ii) let U, V are two open set

T.P $U \cap V$ is open

Let $x \in U \cap V$ T.P x is interier point

$\Rightarrow x \in U$ and $x \in V$

U is open $\Rightarrow \exists\, r_1 > 0 \ni N_{r_1}(x) \subseteq U$

and V is open $\Rightarrow \exists\, r_2 > 0 \ni N_{r_2}(x) \subseteq V$

Let $r = \min\{r_1, r_2\}$

$N_r(x) \subseteq N_{r_1}(x) \subseteq U$

$N_r(x) \subseteq N_{r_2}(x) \subseteq V$

$\therefore N_r(x) \subseteq U \cap V$

$\therefore x$ is an interier point $\Rightarrow U \cap V$ is open

(iii) Let $\{V_i\}_{i \in I}$ $\{I$ is finite or infinite$\}$ be any fimily of open set of X.

Let $x \in \bigcup_{i \in I} V_i \Rightarrow x \in V_i$

Since V_i is open set then there exist positive real number $r > 0 \ni N_r(x) \subseteq V_i$

and $V_i \subseteq \bigcup_{i \in I} V_i \Rightarrow N_r(x) \subseteq \bigcup_{i \in I} V_i$

$\therefore x$ is an interier point of $\bigcup_{i \in I} V_i$

$\Rightarrow \bigcup_{i \in I} V_i$ is open set.

<u>Remark</u>:- The intersection of any infinite Collection of open sets need not to be open.

Corollaray:- let (X,d) be ametric space. Then A open iff $A = \bigcup_{x\in A} N_r(x)$.

proof: $\Leftarrow$ suppose $A = \bigcup_{x\in A} N_r(x)$

T.P A is open

Since $N_r(x)$ is an open set and Union of family op set is open, Then A is open.

$\Rightarrow$ Conversly A is open

T.P $A = \bigcup_{x\in A} N_r(x)$

Since A is open, Then $\forall x \in A \ \exists r>0 \ni N_r(x) \subseteq A$

$\Rightarrow \bigcup_{x\in A} N_r(x) \subseteq A$ --- ①

$\{x\} \subseteq N_r(x) \Rightarrow \bigcup_{x\in A} \{x\} \subseteq \bigcup_{x\in A} N_r(x)$

$A \subseteq \bigcup_{x\in A} N_r(A)$ --- ②

from ① and ② we get $A = \bigcup_{x\in A} N_r(x)$

Definitsion:- let (X,d) be ametric space, $E \subseteq X$ apoint $p \in X$ is Called Limit point of E if every niegbourhood of p $(N_r(p) - \{p\}) \cap E \neq \phi$.

Example:- Let (R,d) be ausuall metric space $A = (2,3]$

2 is Limit point of A becouse every $N_r(2) - \{2\} A \neq \phi$

0 is not Limit point of A becouse $\exists N_r(0) - \{0\} \cap F = \phi$

Afinite set has no limit point

Definition:- let (X,d) be a metric space. A subset A of X is called closed set if every Limit point of A is a point in A.

Example:- Let (R,d) be a usuall metric space

1- $[3,5]$ is closed
let $y \in [3,5]$
Then every $N_r(y)-\{y\} \cap [3,5] \neq \phi$
$\therefore$ every Limit point of $[3,5]$ in $[3,5]$
$\therefore$ $[3,5]$ is closed set.

2- $(3,5]$ is not closed
Since 3 is Limit point of $(3,5]$
but $3 \notin (3,5]$.

Theorem:- If p is a limit point of a set E. Then every niegh of p contians infinitely many point of E.

Proof:- Let N be a niegh of $p \ni N$ contains finite number of point of E
Let $q_1, q_2, \dots, q_n \in N \cap E$
Let $r = \min\{d(p,q_1), d(p,q_2), \dots, d(p,q_n)\}$
Since $d(p,q) > 0 \Rightarrow r > 0$
$\therefore \exists N_r(p)$ is a neigh of $p \ni$
$(N_r(p) - \{p\}) \cap E = \phi$
which is contraduction
becouse p is a Limit point of E
$\therefore N$ contians an infinite number of point of E.

Corollaray :- A finite set has no Limit point

Proof :- Let U be a finite set and U has Limit point is p

$\therefore (N_r(p) - \{p\}) \cap U \neq \Phi$

let $(N_r(p) - \{p\}) \cap U = A$

by theorem above [every neigh of point p contians infinity point of U]

$\Rightarrow$ A is infinity set but $A \subseteq U$ and U is finite set C!

$\therefore$ U has no Limit point.

مهم

Theorem :- A set E is open iff its complement is closed.

proof :- Suppose E is open T.P E^c is closed

let x be a Limit point of E^c

$\therefore \forall$ neigh N of x $(N - \{x\}) \cap E^c \neq \Phi$

There is no neigh of x which is subset of

Thus x is not interier point of E

Since E is open and $x \notin E$

$\Rightarrow x \in E^c$

Conversely $\Leftarrow$ let E^c is closed T.P E is open

let $x \in E \Rightarrow x \notin E^c$

Since E^c is closed

$\Rightarrow x$ is not Limit point of E

$\exists$ neigh N of $x \ni (N - \{x\}) \cap E^c = \Phi$

$\therefore N \subseteq E$

$\therefore x$ is interier point of E

$\therefore E$ is open

Corrollary:- A set F is closed iff F^c is open.

Theorem:- let (X,d) be a metric space. Then
1- X, ϕ are closed set
2- The Union of any two closed set is closed
3- The intersection of any family of closed set is closed

Proof: 1) Since ϕ is open and $X = X - \phi$
$\Rightarrow X$ is closed
Since X is open and $\phi = X - X$
$\Rightarrow \phi$ is closed.

2) let F_1, F_2 are closed set T.p $F_1 \cup F_2$ is closed
Let $F_1 = X - V_1$, $F_2 = X - V_2$
Then V_1 and V_2 are open

Then $F_1 \cup F_2 = (X - V_1) \cup (X - V_2)$
$= X - (V_1 \cap V_2)$
but $V_1 \cap V_2$ is open
$\therefore F_1 \cup F_2$ is closed

3) Let $\{F_i\}_{i \in I}$ be a family of closed set
T.P $\cap F_i$ is closed

$F_i = X - V_i$
$\cap F_i = \cap (X - V_i)$
$= X - \cup V_i$
but $\cup V_i$ is open

$\Rightarrow \cap F_i$ is closed

Corollaray :- ① For any finite collection $G_1, G_2, \dots, G_n$ of open sets $\bigcap_{i=1}^{n} G_i$ is open.

② For any finite collection $F_1, F_2, \dots, F_n$ of closed sets $\bigcup_{i=1}^{n} F_i$ is closed.

proof :- ① let $x \in \bigcap_{i=1}^{n} G_i$

$\Rightarrow x \in G_i \quad i = 1, 2, 3, \dots$

Since G_i is an open set

Then x is an interier point of G_i

$\therefore \exists$ a neigh N_{r_i} of $x \ni N_{r_i} \subseteq G_i$

Thus $N_{r_1}(x) \subseteq G_1$, $N_{r_2}(x) \subseteq G_2$, $\dots \longrightarrow$

$N_{r_n}(x) \subseteq G_n$

Let $r = \min\{r_1, r_2, \dots, r_n\}$

$\therefore N_r(x) \subseteq N_{r_i}(x) \quad i = 1, 2, \dots$

$\therefore N_r(x) \subseteq \bigcap_{i=1}^{n} G_i$

$\therefore x$ is an interier point of $\bigcap_{i=1}^{n} G_i$

$\therefore \bigcap_{i=1}^{n} G_n$ is an open set.

2 Since F_i is closed $\quad i = 1, 2, \dots, n$

$\Rightarrow F_i^c$ is open

$\therefore \bigcap_{i=1}^{n} F_i^c$ is open $\quad$ (by 1)

$\therefore \left(\bigcap_{i=1}^{n} F_i^c\right)^c$ is closed

$\Rightarrow \bigcup_{i=1}^{n} (F_i^c)^c$ is closed

$\Rightarrow \bigcup_{i=1}^{n} F_i$ is closed.