Definition ✓

• Let (X,d) be a metric space, $A \subseteq X$, the set of all Limit points of A. is called derived set and denoted by $D(A)$. or A'

The closure of a set A is the set $A \cup A'$ and denoted by $\overline{A}$ or $cL(A)$.

Examples :- Let $(\mathbb{R}, d)$ be usual metric space.

1. $A = (0,1)$

1. $\forall \ y \in (0,1) \Rightarrow (N_r(y) - \{y\}) \cap A \neq \phi. \Rightarrow y$ is Limit point
2. if $y < 0 \Rightarrow \exists r > 0 \ni (N_r(y) - \{y\}) \cap (0,1) = \phi.$
3. if $y > 1 \Rightarrow \exists r > 0 \ni (N_r(y) - \{y\}) \cap (0,1) = \phi.$ $\Big\}$ y is not Limit point
4. if $y = 0 \Rightarrow \exists r > 0 \ni (N_r(0) - \{0\}) \cap (0,1) \neq \phi.$
5. if $y = 1 \Rightarrow \exists r > 0 \ni (N_r(1) - \{1\}) \cap (0,1) \neq \phi.$ $\Big\}$ y is Limit point

$\therefore$ if $A = (0,1) \Rightarrow A' = [0,1]$.

$\Rightarrow \overline{A} = [0,1] \cup (0,1) = [0,1]$.

2. $B = [3,5]$

Case 1: if $y \in [3,5] \Rightarrow (N_r(y) - \{y\}) \cap B \neq \phi.$

$\therefore y$ is Limit point of B.

every point of B is a Limit in B.

Remark : A subset A of a metric space (X,d) is said to be closed iff $A' \subseteq A$.

i.e $A \subseteq X \Leftrightarrow A' \subseteq A$.

Ex, Let (X,d) be a metric space.

1. $A = (2,3)$

   $A' = [2,3]$

   since $A' \nsubseteq A \Rightarrow A$ is not closed set.

2. $B = [3,10]$

   $B' = [3,10]$

   since $B' \subseteq B \Rightarrow B$ is closed set

Theorem :- Let (X,d) be a metric space and $E \subseteq X$, then

1. $\bar{E}$ is closed
2. $E = \bar{E}$ iff E is closed.
3. $\bar{E} \subseteq F$ for every closed set $F \ni E \subseteq F$.

proof Exc.

Definition :-

Let (X, d) be a metric space and $E \subseteq X$ is called dense in X if $\bar{E} = X$

Example: Let $(\mathbb{R}, d)$ be usual metric space.

The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$ since $\overline{\mathbb{Q}} = \mathbb{R}$.

Definition:- Let (X,d) be a metric space, For each $x \in X$ and $A \subseteq X$, we have

$$d(x,A) = g.l.b\{d(x,a) ; a \in A\}.$$

$d(x,A)$ is called the distinct between the point x and the set A.

Definition:- Let (X,d) be a metric space, For any two subset A, B of X, we have.

$$d(A,B) = g.l.b\{d(a,b) : a \in A, b \in B\}$$

$d(A,B)$ is called the distant between the set A and B.

Examples:- 1. Find $d(5,[10,20])$

$$d(x,A) = g.l.b\{d(x,a) : a \in A\}$$

$$d(5,[10,20]) = g.l.b\{d(5,a) : a \in [10,20]\}.$$

$$\therefore = g.l.b[5,15] = 5.$$

2. Find $d([7,10],(13,20])$

$$= g.l.b\{d(a,b) : a \in [7,10], b \in (13,20]\}$$

$$= 3$$

Definition Let (X, d) be a metric space and $E \subseteq X$ is called bounded set if there is a real number M and $q \in X$ such that $d(p, q) < M$ for all $x \in E$

Ex/ Let $(\mathbb{R}, d)$ be usual metric space $E = \{3, 4, 5, 6\}$

Then $\exists\, M = 12$, $q = 7$

$d(p, 7) < 12$, $\forall\, p \in E$.

Ex/ $E = (-1, 7)$

Then $\exists\, M = 10$, $q = 8$

$d(q, 8) < 10$, $\forall\, p \in E$.