# (Numerical sequence)

## convergent sequence.

**Definition:-** let $(x,d)$ be a metric space and $a \in X$ a sequence $\langle a_n \rangle$ is said to be converge to $a$ if for each $\epsilon > 0$, there exists a positive integer $N$ $\ni d(a_n,a) < \epsilon \quad \forall\, n \geq N$.

if $\langle a_n \rangle$ does not converges then it is called diverges.

**Remarks:-**
1- $\langle a_n \rangle$ converges to $a$ also means $a$ is limit point of $\langle a_n \rangle$ and we written $a_n \to a$ or $\lim_{n \to \infty} a_n = a$.

2- The set $\{a_1, a_2, a_3, \dots\}$ is called the range of $\langle a_n \rangle$

**Definition:-** let $(x,d)$ be a metric space and $a \in X$.
A sequence $\langle a_n \rangle$ is called bounded if its range is bounded set.

**Example(1):-**
let $(R,d)$ be usual metric space, show that $\langle \frac{1}{n} \rangle$ converges to $0$.

proof:- 1. let $\epsilon > 0$ be given.

2. T.P. $\exists$ positive integer $N \ni d(a_n,a) < \epsilon \quad \forall\, n \geq N$.

i.e $\exists$ positive integer $N \ni |a_n - a| < \epsilon. \quad \forall\, n \geq N$.

" " " $|\frac{1}{n} - 0| < \epsilon$ "

3. let $N$ be smallest positive integer $\ni N > \frac{1}{\epsilon}$

since $n \geq N > \frac{1}{\epsilon} \Rightarrow \frac{1}{n} \leq \frac{1}{N} < \epsilon$

$$\Rightarrow |\frac{1}{n}| < \epsilon \Rightarrow |\frac{1}{n} - 0| < \epsilon$$

$$\Rightarrow d(a_n, a) < \epsilon$$

$\therefore \langle a_n \rangle$ is converges to $0$

The sequence $\langle \frac{1}{n} \rangle$ is bounded

since the range of $\langle \frac{1}{n} \rangle$ is $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.

it is bounded $\Rightarrow$ The sequence $\langle \frac{1}{n} \rangle$ is bounded sequence.

**Example ②:-** show that the sequence $\langle \frac{1}{n+1} \rangle$ converges to $0$.

proof :- 1. let $\epsilon > 0$ be given

2. T.P $\exists$ positive integer $N \ni |\frac{1}{n+1} - 0| < \epsilon \quad \forall n \geq N$.

3. let $N$ be smallest positive integer $\ni N > \frac{1}{\epsilon} - 1$.

since $n \geq N > \frac{1}{\epsilon} - 1 \Rightarrow n > \frac{1}{\epsilon} - 1 \Rightarrow n+1 > \frac{1}{\epsilon}$

$$\Rightarrow \frac{1}{n+1} < \epsilon \Rightarrow |\frac{1}{n+1} - 0| < \epsilon \Rightarrow d(a_n, a) < \epsilon \quad \forall n \geq N$$

$\therefore \langle \frac{1}{n+1} \rangle$ is converges to $0$.

The sequence $\langle \frac{1}{n+1} \rangle$ is bounded, since the range of $\langle \frac{1}{n+1} \rangle$ is $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ which is bounded set

**Theorem :-** let $\langle a_n \rangle$ be a sequence in a metric space $(X,d)$, then $\langle a_n \rangle$ converges to $a \in X$ iff every nigh. of $a$ contains all elements $a_n$ expect a finit set.

**proof :-** $\Rightarrow$ let $\langle a_n \rangle$ converges to $a \in X$ and let $U$ be a nigh of $a$.

let $\epsilon > 0$ be the radius of $U$.

since $\langle a_n \rangle$ is conv. to $a$

$\Rightarrow \exists$ a positive integer $N \ni d(a_n, a) < \epsilon$ for all $n \geq N$.

Then $\forall n \geq N$, $a_n \in U$. and $a_1, a_2, \ldots, a_{n-1} \notin U$.

Then the set $\{a_1, a_2, \ldots, a_n\}$ is finite.

$\Leftarrow$ suppose every nigh $U$ contains all element $a_n$ expect a finite set of them., T.P. $\langle a_n \rangle$ converges to $a$.

let $\epsilon > 0$ be given, and $U = \{b \in X \cdot d(a,b) < \epsilon\}$

$\therefore$ $U$ is a nigh of $a$ with radius $\epsilon$.

Then by assumption, $\exists$ positive integer $N \ni$ $a_1, a_2, \ldots, a_{n-1} \notin U$. and $a_n \in U$, $\forall n \geq N$.

$\therefore$ $\exists$ positive integer $N \ni d(a_n, a) < \epsilon$ $\forall n \geq N$.

$\therefore$ $\langle a_n \rangle$ is converges to $a$.

**Theorem** :- Let $\langle a_n \rangle$ be a sequnce in a metric space $(X,d)$, if $a_1, a_2 \in X$, and $\langle a_n \rangle$ converges to $a_1, a_2$ Then $a_1 = a_2$.

**proof** :- Since $\langle a_n \rangle$ converges to $a_1$

$\Rightarrow \exists$ positive integer $N_1 \ni d(a_n, a_1) < \frac{\epsilon}{2} \quad \forall n \geq N_1$

Since $\langle a_n \rangle$ converges to $a_2$

$\Rightarrow \exists$ positive integer $N_2 \ni d(a_n, a_2) < \frac{\epsilon}{2} \quad \forall n \geq N_2$

let $N = \max\{N_1, N_2\}$.

$\therefore d(a_n, a_1) < \frac{\epsilon}{2}, \quad d(a_n, a_2) < \frac{\epsilon}{2} \quad \forall n \geq N$.

$$d(a_1, a_2) \leq d(a_1, a_n) + d(a_n, a_2)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore d(a_1, a_2) < \epsilon$

$\Rightarrow d(a_1, a_2) = 0 \Rightarrow a_1 = a_2$.

**Definition** :- Let $(X,d)$ be a metric space and let $y$ be a fixed point in $X$., a subset $E$ of $X$ is called bounded if $\exists$ a positive real number $M \ni$ $d(x,y) \leq M$ for all point $x \in X$.

**Theorem** :- Let $\langle a_n \rangle$ be a sequence in a metric space $(X,d)$ converges to $a$, then $\langle a_n \rangle$ is bounded.

**proof** :- let $\epsilon = 1$

since $\langle a_n \rangle$ converges to $a$

$\Rightarrow \exists$ positive integer $N \ni d(a_n, a) < 1$.

Let $M = \max\{1, d(a_1,a), d(a_2,a), d(a_3,a), \dots d(a_{n-1},a)\}$

$\Rightarrow d(a_n, a) < M$, for $n = 1,2,3,\dots,n-1$

the range of $\langle a_n \rangle$ is bounded.

$\therefore \langle a_n \rangle$ is bounded (by definition of bounded seq.)

**Theorem** :- let $\langle a_n \rangle$ be a seq. in a metric space $(X,d)$ if $E \subseteq X$, and $a$ is a Limit point of $E$, then there is a seq. $\langle a_n \rangle$ in $E \ni \lim_{n\to\infty} a_n = a$

**proof** :- $\forall n \in \mathbb{Z}^+$, $\exists N_{\frac{1}{n}}(a)$ (neighbourhood of $a$) s.t $N_{\frac{1}{n}}(a) - \{a\}) \cap E \neq \Phi$. (since $a$ is a Limit point of $E$)

$\Rightarrow a_n \in E$, $\forall n = 1,2,3,\dots$

$\Rightarrow \langle a_n \rangle$ is a sequence in $E$.

$\therefore a_n \in N_{\frac{1}{n}}(a) \quad \forall n$.

$\Rightarrow d(a_n, a) < \frac{1}{n} \quad \forall n$.

let $\epsilon > 0 \Rightarrow \exists$ positive integer $N \ni N > \frac{1}{\epsilon}$

if $n \geq N \Rightarrow n > \frac{1}{\epsilon} \Rightarrow \frac{1}{n} < \epsilon$.

$\therefore d(a_n, a) < \epsilon \quad \forall n \geq N$.

$\Rightarrow \lim a_n = a$

Theorem :-

suppose $\langle a_n \rangle$, $\langle b_n \rangle$ are two real sequ[ences]

and $\lim_{n\to\infty} a_n = a$ , $\lim_{n\to\infty} b_n = b$ , then

ⓐ $\lim_{n\to\infty} (a_n + b_n) = a + b$

ⓑ $\lim_{n\to\infty} (K \cdot a_n) = K.a$ , $\lim_{n\to\infty} (K + a_n) = K + a$

ⓒ $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$ , $(a_n \neq 0, a \neq 0$, for $n = 1, 2, \ldots)$

ⓓ $\lim_{n\to\infty} (a_n \cdot b_n) = a \cdot b$.

proof :- ⓐ let $\epsilon > 0$ be given.

Since $\langle a_n \rangle \to a$ $(\lim_{n\to\infty} a_n = a)$

$\Rightarrow \exists$ positive integer $N_1 \ni |a_n - a| < \frac{\epsilon}{2}$ $\forall n \geq N_1$

Since $\lim_{n\to\infty} b_n = b$

$\Rightarrow \exists$ positive integer $N_2 \ni |b_n - b| < \frac{\epsilon}{2}$ $\forall n \geq N_2$.

let $N = \max\{N_1, N_2\}$.

$$\therefore |(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)|$$
$$\leq |a_n - a| + |b_n - b|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$\therefore |(a_n + b_n) - (a + b)| < \epsilon$.

(6) $\lim_{n\to\infty} (K a_n) = K a$.

proof:- let $\epsilon > 0$ be given

since $\lim_{n\to\infty} a_n = a \Rightarrow \exists$ positive integer $N$

$\ni |a_n - a| < \frac{\epsilon}{|K|} \quad \forall n \geq N, \; K \neq 0$.

$\therefore |Ka_n - Ka| = |K(a_n - a)| = |K| . |a_n - a|$

$< |K| \frac{\epsilon}{|K|} = \epsilon$.

$\therefore |K a_n - Ka| < \epsilon$.

proof $\lim_{n\to\infty} (K + a_n) = K + a$

let $\epsilon > 0$ be given

since $\lim_{n\to\infty} a_n = a \longrightarrow \exists$ positive integer $N$

$\ni |a_n - a| < \epsilon \quad \forall n \geq N$.

$\Rightarrow |(a_n + K) - (a + K)| = |a_n + K - a - K| = |a_n - a| < \epsilon$

$\forall n \geq N$.

$\therefore |(a_n + K) - (a + K)| < \epsilon$

$\therefore \lim_{n\to\infty} (K + a_n) = K + a$.

proof ⓓ $\lim_{n\to\infty} (a_n \cdot b_n) = a \cdot b$

since $\langle a_n \rangle$, $\langle b_n \rangle$ are converges sequance.

$\therefore$ $\langle a_n \rangle$, $\langle b_n \rangle$ are bounded sequance, then by def.

$\Rightarrow \exists$ positive real number $M_1 > 0$ and $M_2 > 0$

s.t $|a_n| \leq M_1$, $|b_n| \leq M_2$.

let $M = \max\{M_1, M_2\}$.

since $\langle a_n \rangle$ conv. to $a$.

$\Rightarrow \exists$ a positive integer $N_1 \ni |a_n - a| < \frac{\epsilon}{2M_1} \quad \forall n \geq N_1$

since $\langle b_n \rangle$ conv. to $b$.

$\Rightarrow \exists$ a positive integer $N_2 \ni |b_n - b| < \frac{\epsilon}{2M_2} \quad \forall n \geq N_2$

Let $N = \max\{N_1, N_2\}$.

$$
\begin{aligned}
\therefore |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\
&= |a_n(b_n - b) + b(a_n - a)| \\
&\leq |a_n||b_n - b| + |b||a_n - a| \\
&< M_1 \cdot \frac{\epsilon}{2M_1} + M_2 \cdot \frac{\epsilon}{2M_2} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

$\therefore |a_n b_n - ab| < \epsilon \quad \forall n \geq N$.

$\lim_{n\to\infty} a_n b_n = ab$.

**Definition :-** Let $(X,d)$ be a metric space a sequence $\langle a_n \rangle$ is called cauchy sequence if for every $\epsilon > 0$, there is positive integer $N$, such that

$d(a_n, a_m) < \epsilon \quad \forall n \geq N$ and $m \geq N$.

**Example :-** let $(\mathbb{R}, d)$ be the usual metric space, show that a sequence $\langle \frac{1}{n} \rangle$ is cauchy.

**Solution :-** 1. let $\epsilon > 0$ be given

2. let $N$ be the smallest positive integer $\ni N > \frac{2}{\epsilon}$

$\forall n \geq N, m \geq N.$

$\because n \geq N > \frac{2}{\epsilon} \Rightarrow \frac{1}{n} \leq \frac{1}{N} < \frac{\epsilon}{2} \Rightarrow \frac{1}{n} < \frac{\epsilon}{2} \Rightarrow |\frac{1}{n}| < \frac{\epsilon}{2}$

by the same way

$m \geq N > \frac{2}{\epsilon} \Rightarrow \frac{1}{m} \leq \frac{1}{N} < \frac{\epsilon}{2} \Rightarrow \frac{1}{m} < \frac{\epsilon}{2}$

$\Rightarrow |\frac{1}{m}| < \frac{\epsilon}{2}$

$\therefore d(a_n, a_m) = |\frac{1}{n} - \frac{1}{m}| \leq |\frac{1}{n}| + |\frac{1}{m}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\therefore d(a_n, a_m) < \epsilon$

$\therefore \langle \frac{1}{n} \rangle$ is cauchy sequence.

**Example:-** Let $(\mathbb{R}, d)$ be a usual metric space, show that the sequance $\left\langle \frac{1}{n+1} \right\rangle$ is cauchy.

**Solution:-** 1. let $\epsilon > 0$ be given

2. let $N$ be smallest positive integer $\ni N > \frac{2}{\epsilon} - 1$
$\forall n \geq N, m \geq N$.

since $n \geq N > \frac{2}{\epsilon} - 1$

$$\Rightarrow n > \frac{2}{\epsilon} - 1 \Rightarrow n+1 > \frac{2}{\epsilon} \Rightarrow \frac{1}{n+1} < \frac{\epsilon}{2}$$

$$\Rightarrow \left| \frac{1}{n+1} \right| < \frac{\epsilon}{2}.$$

by the same way $\left| \frac{1}{m+1} \right| < \frac{\epsilon}{2}$.

$$\therefore d(a_{n+1}, a_{m+1}) = \left| \frac{1}{n+1} - \frac{1}{m+1} \right| \leq \left| \frac{1}{n+1} \right| + \left| \frac{1}{m+1} \right|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\therefore d(a_{n+1}, a_{m+1}) < \epsilon$$

$\therefore \left| \frac{1}{n+1} - \frac{1}{m+1} \right| < \epsilon \Rightarrow \left\langle \frac{1}{n+1} \right\rangle$ is Cauchy sequance.

**Theorem:-** In any metric space, every convergent sequnce is a cauchy sequnce.

**proof:-** let $(X, d)$ be a metric space.
let $\langle a_n \rangle$ be convergent sequance to $a$
let $\epsilon > 0$ be given.

since $\langle a_n \rangle$ converges to $a$

$\Rightarrow \exists$ positive integer $N \ni d(a_n, a) < \frac{\epsilon}{2} \quad \forall n \geq N$.

$\therefore d(a_n, a_m) \leq d(a_n, a) + d(a, a_m)$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore d(a_n, a_m) < \epsilon$

$\therefore \langle a_n \rangle$ is a Cauchy sequence

Remark ①: - The converse of above theorem is not true in general

Example. Let $(\mathbb{R} - \{0\}, d)$ be a usual metric space. The sequence $\langle \frac{1}{n} \rangle$ is Cauchy sequence in a metric space $(\mathbb{R} - \{0\}, d)$, but is not convergent sequence.

since $\langle \frac{1}{n} \rangle$ converges to zero and $0 \notin (\mathbb{R} - \{0\}, d)$

Therefore $\langle \frac{1}{n} \rangle$ is diverges.

Remark ② A bounded sequence is not necessary to be convergent sequence

Example. (Exe)

Definition :- 1. A sequence $\langle a_n \rangle$ of real numbers is called increasing sequence if $a_n < a_{n+1}$ $\forall n$

2. A sequence $\langle a_n \rangle$ of real numbers is called decreasing sequence if $a_n > a_{n+1}$ $\forall n$.

3. A sequence $\langle a_n \rangle$ is called monotonic sequence if $\langle a_n \rangle$ either increasing or decreasing sequence.

Example :-

1. The sequence $\langle \frac{1}{n} \rangle$ is decreasing seq.

$n+1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n}$

$\Rightarrow a_{n+1} < a_n$ $\forall n$.

2. The sequence $\langle n^2 \rangle$ is increasing seq.

$n^2 < (n+1)^2 \Rightarrow a_n < a_{n+1}$ $\forall n$.

$\langle 1, 4, 9, 16, \ldots \rangle$

**Theorem** :- 1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1.$

3. $\lim_{n\to\infty} \sqrt[n]{p} = 1 \quad (p > 0).$

4. $\lim_{n\to\infty} p^n = 0 \quad$ if $(|p| < 1.)$

5. $\lim_{n\to\infty} (1 + \frac{p}{n})^n = e^p$ (where $p$ is any real numbers)

proof ② let $a_n = \sqrt[n]{n} = (n)^{1/n}$

$\ln a_n = \ln (n)^{1/n} = \frac{1}{n} \ln (n).$

$\therefore \lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{1}{n} \ln (n) = \lim_{n\to\infty} \frac{\ln(n)}{n} = 0$ (by part ①)

$\therefore e^{\lim_{n\to\infty} \ln a_n} = e^0$

$\Rightarrow e^{\ln (\lim_{n\to\infty} a_n)} = e^0$

$\Rightarrow \lim_{n\to\infty} a_n = 1.$

$\therefore \lim_{n\to\infty} \sqrt[n]{n} = 1.$