3-Nil Radical Ideal

Definition:- (**Nil Radical**) معدوم ألجذر

Let (I,+ , .)be an ideal of the ring (R, +, .) . Then **the nil radical** of ideal **I** denoted by $\sqrt{\mathbf{I}}$,and is defined by:

$\sqrt{I} = \{ r \in R : r^n \in I, \text{ for some } n \in Z_+ \}$.

(i.e.) , if $r \in \sqrt{I} \Leftrightarrow r^n \in I$, for some $n \in Z_+$

Example: Show that $\sqrt{I} = \{ r \in R : r + I \text{ is nilpotent in } R/I \}$

Sol.: $\sqrt{I} = \{ r \in R : r^n \in I, \text{ for some } n \in Z^+ \}$

$= \{ r \in R : r^n + I = I, \text{ for some } n \in Z^+ \}$

$= \{ r \in R : (r + I)^n = I, \text{ for some } n \in Z^+ \}$

$= \{ r \in R : r + I \text{ is nilpotent in } R/I \}$

Remark :-

(1) If ((n) ,+, .) is a principal ideal of the ring (Z, + , .) ,n > 1 , such that $n = P_1^{k_1} . P_2^{k_2} \ldots\ldots\ldots P_r^{k_r}$, where $P_1, P_2, \ldots, P_r$ are distinct prime number and $k_1, k_2, \ldots\ldots, k_r$ .Then

$$\sqrt{(n)} = \sqrt{(P_1^{k_1} . P_2^{k_2} \ldots\ldots P_r^{k_r})} = (P_1 . P_2 \ldots\ldots P_r) \;..$$

(2) $\sqrt{(P)} = (P)$, where P is a prime number .

Example :- In a ring of (Z , + , .) , find

(1) $\sqrt{(12)}$

(2) $\sqrt{(36)}$

(3) $\sqrt{(8)}$

(4) $\sqrt{(17)}$

Solution(1) :- To find $\sqrt{(12)}$?

$\because$ 12=4. 3 = 2 .2 .3 = $2^2.3$

$\Longrightarrow \sqrt{(12)} = \sqrt{(2^2.3)} = (2.3) = (6) = \{0, \pm 6, \pm 12, \pm 18, \dots\}$

Solution(2) :- To find $\sqrt{(36)}$?

$\because$ 36=4·9=2.2.3.3= $2^2.3^2$

$\Longrightarrow \sqrt{(36)} = \sqrt{(2^2.3^2)} = (2.3) = (6) = \{0, \pm 6, \pm 12, \pm 18, \dots\}$

Solution(3) :- To find $\sqrt{(8)}$?

$\because$ 8=2.4=$2^3 \Longrightarrow \sqrt{(8)} = \sqrt{(2^3)} = (2) = \{0, \pm 2 \ \pm 4, \pm 6, \dots\}$

Solution (4):- To find $\sqrt{(17)}$?

$\because$ 17 is a prime number

$\Longrightarrow \sqrt{(17)} = (17) = \{0, \pm 17, \pm 34, \pm 51, \dots\}$

Theorem (2-20):- If (I,+ , .)is an ideal of the ring (R, +, .) . Then

1- $I \subseteq \sqrt{I}$.

2- $(\sqrt{I}, +, .)$ is an ideal of a ring R is called nil radical ideal .

Proof(1):-

المعطى(I,+ , .)is an ideal of the ring (R, +, .)

< T.P. $I \subseteq \sqrt{I}$ >?

Let $a \in$ I

$\Longrightarrow \underbrace{a.a.a \dots a}_{\text{n-times}} \in$ I (since I is an ideal of R)

$\Longrightarrow a^n \in I \quad \Longrightarrow a \in \sqrt{I}$

$\therefore I \subseteq \sqrt{I}$.

Proof(2):-

T.P. , $(\sqrt{I}\ ,+,.)$ is an ideal of a ring R

(i) Let $a, b \in \sqrt{I}$ < **T-P , $a - b \in \sqrt{I}$** >

$\because a \in \sqrt{I} \Longrightarrow a^{m} \in I$, for some $m \in Z^{+}$

$\because b \in \sqrt{I} \Longrightarrow b^{n} \in I$, for some $n \in Z^{+}$

Since , I is an ideal of R

$\Longrightarrow a^{m} - b^{n} \in I$

$\Longrightarrow (a-b)^{m+n} \in I$ (by Theorem , $a^{m} - b^{n} = (a-b)^{m+n}$)

$\Longrightarrow a - b \in \sqrt{I}$ (by definition of nil radical of ideal I)

(ii) Let $r \in R$ and $a \in \sqrt{I}$

$\therefore a^{n} \in I$, for some $n \in Z_{+}$

$\because r \in R$, since R is a ring

$\Longrightarrow \underbrace{r.r.r\ldots r}_{n-\text{times}} \in R \Longrightarrow r^{n} \in R$

$\Longrightarrow a^{n}.r^{n} \in I$ and $r^{n}.a^{n} \in I$ (I is an ideal)

$\Longrightarrow (a.r)^{n} \in I$ and $(r.a)^{n} \in I$

$\Longrightarrow a.r \in \sqrt{I}$ and $r.a \in \sqrt{I}$

Therefore $(\sqrt{I}\ ,+,.)$ is an ideal of R .

Theorem(2-21):-

If (I,+, .)and (J,+,.)are two ideals of the ring (R, +, .) . Then

1- $\sqrt{\sqrt{I}} = \sqrt{I}$

2- $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$

3 – If $I^k \subseteq J$, for some $k \in Z_+$. Then $\sqrt{I} \subseteq \sqrt{J}$ (H-W واجب)

Proof (1) :-

$< \text{T-P}, \sqrt{\sqrt{I}} = \sqrt{I} >$ → **(i)** $\sqrt{\sqrt{I}} \subseteq \sqrt{I}$; **(ii)** $\sqrt{I} \subseteq \sqrt{\sqrt{I}}$

(i) $<$ **T-P ,** $\sqrt{\sqrt{I}} \subseteq \sqrt{I}$ $>$?

Let $r \in \sqrt{\sqrt{I}}$

$\Longrightarrow r^n \in \sqrt{I}$, for some $n \in Z^+$ (by def. nil radical of ideal)

$\Longrightarrow (r^n)^m \in I$, for some $m \in Z^+$ (by def. nil radical of ideal)

$\Longrightarrow r^{n.m} \in I$ (since $n , m \in Z^+ \Longrightarrow n.m \in Z^+$)

$\Longrightarrow r \in \sqrt{I}$ by(def. nil radical of ideal)

$\Longrightarrow \sqrt{\sqrt{I}} \subseteq \sqrt{I}$.

(ii) $<$ **T-P** $\sqrt{I} \subseteq \sqrt{\sqrt{I}}$ $>$?

By Th.(2-20)step-2-

$\Longrightarrow \sqrt{I}$ is an ideal of the ring R

Also, by using Th.(2-20)step-1- we get

$\Longrightarrow \sqrt{I} \subseteq \sqrt{\sqrt{I}}$

Therefore , by (i) and (ii) we get $\sqrt{\sqrt{I}} = \sqrt{I}$. □

Proof(2):- $< $ **T-P,** $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J} >$

(i.e.) T-P (**i**) $\sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J}$

(**ii**) $\sqrt{I} \cap \sqrt{J} \subseteq \sqrt{I \cap J}$

(i) $<$ **First, T-P** $\sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J} > ?$

Let $r \in \sqrt{I \cap J}$

$\Longrightarrow r^n \in I \cap J$, for some $n \in Z^+$ (by def. nil radical of ideal)

$\Longrightarrow r^n \in I$ and $r^n \in J$

$\Longrightarrow r \in \sqrt{I}$ and $r \in \sqrt{J}$ (by def. nil radical of ideal)

$\Longrightarrow r \in \sqrt{I} \cap \sqrt{J}$

Thus, $\sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J}$

(ii) $<$ **Now, T-P,** $\sqrt{I} \cap \sqrt{J} \subseteq \sqrt{I \cap J} > ?$

Let $r' \in \sqrt{I} \cap \sqrt{J}$

$\Longrightarrow r' \in \sqrt{I}$ and $r' \in \sqrt{J}$

$\Longrightarrow r'^m \in I$ and $r'^n \in J$ (for some $m, n \in Z^+$)

$\because I, J \subseteq R \Longrightarrow r'^m, r'^n \in R$

$\because r'^m \in I$ and $r'^n \in R$ and since I is an ideal of R

$\Longrightarrow r'^m . r'^n \in I \Longrightarrow r'^{m+n} \in I \cdots *$

Also, $\because r'^n \in J$ and $r'^m \in R$ and since J is an ideal of R

$\Longrightarrow r'^m . r'^n \in J \Longrightarrow r'^{m+n} \in J \cdots **$

$\therefore$ by $*$ and $**$ we get $\Longrightarrow r'^{m+n} \in I \cap J \Longrightarrow r' \in \sqrt{I \cap J}$

$\therefore \sqrt{I} \cap \sqrt{J} \subseteq \sqrt{I \cap J}$

Therefore, by i, ii we get $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$. □

Example :- In a ring of (Z,+, .) . Find

1- $\sqrt{\sqrt{(18)}}$ 2– $\sqrt{(36)\cap(3)}$

3- $\sqrt{\sqrt{(32)}}\ \cap\ \sqrt{(6)}$ 4- $\sqrt{\sqrt{(40)\cap(10)}}$ (واجب)

Solution(1):- To find $\sqrt{\sqrt{(18)}}$?

$\because\ \sqrt{\sqrt{(18)}} = \sqrt{(18)}$ (by th . (2-21)step-1- $\sqrt{\sqrt{I}} = \sqrt{I}$)

And , since 18= 2.9=2.3.3=2.3^2

$\Longrightarrow \sqrt{\sqrt{(18)}} = \sqrt{(18)} = \sqrt{(2.3^2)} = (2.3)=(6)=\{\,0\,,\pm6,\pm12,\ldots\}$

Solution(2):- < To find $\sqrt{(36)\cap(3)}$ > ?

By Th.(2-21)step-2- $\sqrt{I\cap J} = \sqrt{I}\cap\sqrt{J}$

$\Longrightarrow \sqrt{(36)\cap(3)} = \sqrt{(36)}\cap\sqrt{(3)}$

$\because$ 36= 6 .6 = 2.3 .2 .3 = $2^2.3^2$

$\Longrightarrow \sqrt{(36)} = \sqrt{(2^2.3^2)} = (2.3) = (6) = \{0,\pm6,\pm12,\ldots\}$

And ,

$\Longrightarrow \sqrt{(3)} = (3) = \{\,0,\pm3,\pm6,\ldots\}$

$\Longrightarrow \sqrt{(36)\cap(3)} = \sqrt{(36)}\cap\sqrt{(3)} = (6)\cap(3)$

$= \{0,\pm6\ ,\pm12,\ldots\}\cap\{0,\pm3,\pm6,\ldots\}$

$=\{0,\pm6\ ,\pm12,\ldots\}=(6)$

Solution(3): To find $\sqrt{\sqrt{(32)}} \cap \sqrt{(6)}$?

By (Th . (2-21)step-1-) $\Longrightarrow$ $\sqrt{\sqrt{I}} = \sqrt{I} \Longrightarrow \sqrt{\sqrt{(32)}} = \sqrt{(32)}$

$\because 32 = 8.4 = 2.2.2.2.2 = 2^5$

$\Longrightarrow \sqrt{(32)} = \sqrt{(2^5)} = (2) = \{0, \pm 2, \pm 4, \pm 6, \dots\}$

$\because 6 = 2.3$

$\Longrightarrow \sqrt{(6)} = \sqrt{(2.3)} = (2.3) = (6) = \{0, \pm 6, \pm 12 \dots\}$

$\Longrightarrow \sqrt{\sqrt{(32)}} \cap \sqrt{(6)} = \sqrt{(32)} \cap \sqrt{(6)} = (2) \cap (6)$

$= \{0, \pm 6, \pm 12 \dots\}.$

Definition:- (**Primary Ideal** المثالي الابتدائي)

An ideal (I,+, .) of the ring (R , + , .) is called **Primary ideal** if $\forall a, b \in R$ such that $a . b \in I$, implies that if $a \notin I$, then $b^n \in I$ **or** if $b \notin I$ then $a^n \in I$, for some $n \in Z_+$.

Proposition:- Every prime ideal is a primary ideal , but is not converse .

Proof:- Let (I,+,.) be a prime ideal of the ring (R,+, .)

< T-P , I is a primary ideal >?

من الفرض (I,+,.) be a prime ideal of the ring (R,+, .)

$\Longrightarrow \forall a, b \in R$ S . t , $a.b \in I$ and if $a \notin I \Longrightarrow b \in I$.

$\Longrightarrow \underbrace{b.b \ldots \ldots b}_{n-\text{times}} \in I$ (since I is an ideal of R)

$\Longrightarrow b^n \in I$

Therefore , I is a primary ideal . □

Remark: The converse of above proposition is not true in general.

عكس القضية ليس من الضروري صحيح و هذا واضح في المثال التالي هو مثالي ابتدائي ولكن ليس مثالي أولي

Example: Give an example of a primary ideal which is not prime ideal.

Solution :-

In a ring $(Z_{12}, +_{12}, ._{12})$ an ideal $I=(\bar{4})=\{\bar{0}, \bar{4}, \bar{8}\}$ **is not prime ideal**

Since , $\exists\ \bar{2}, \bar{6} \in Z_{12}$ such that $\bar{2}._{12}\bar{6} = \bar{0} \in (\bar{4})$, but $\bar{2} \notin (\bar{4})$ and $\bar{6} \notin (\bar{4})$

But $\bar{2} \notin (\bar{4}) \Longrightarrow \bar{2}^2 \in (\bar{4})$ also $\bar{2}^3 \in (\bar{4})$, where $2,3 \in Z^+$

Hence , $((4), +_{12}, ._{12})$ **is a primary ideal.**

Definition:- (**The Radical of Ideal**) المثالي الجذري

The radical ideal of a ring (R , +, .) is denoted by **rad.(R)** ,where rad.(R) is the set :

rad.(R) = ∩ { M: M is a maximal ideal of R } .

. R هو تقاطع كل المثاليات العظمى في حلقة = rad.(R)

Remark :

If R is field , then rad.(R) = ({0} , + , .) .

Example : - Find

-**A-** rad.(Z_{12}) , rad.(Z_{11}) , rad.(Q) , rad.(Z_{20}) , rad.(Z_{60})

-B- rad.(Z_{72}) , rad.(Z_{29}) , rad.(Z_{40}) (واجب)

Solution (1):- To find rad.(Z_{12})

rad.(Z_{12}) = ∩ { M: M is a maximal ideal of a ring Z_{12} } .

∵ The maximal ideals in Z_{12} are : **12=2(6)=3(4)**

1- $((\bar{2}) ,+_{12}, \cdot_{12})$

2- $((\bar{3}),+_{12} , \cdot_{12})$

∴ rad.(Z_{12})=$(\bar{2})$∩ $(\bar{3})$={ $\bar{0}, \bar{2} , \bar{4}, \bar{6}, \bar{8}, \overline{10}$ } ∩ { $\bar{0}, \bar{3}, \bar{6}, \bar{9}$ }={$\bar{0}, \bar{6}$ }=$(\bar{6})$.

Solution (2):- To find rad.(Z_{11}) ?

rad.(Z_{11}) = ∩ { M: M is a maximal ideal of a ring Z_{11} } .

∵ Z_{11} is a field ⟹ the only maximal ideal in Z_{11} is ((0) ,+ , .)

⟹ rad.(Z_{11}) = ({0} , $+_{11}$, $\cdot_{11}$) .

Solution (3):-

To find rad.(Q) ?

rad.(Q) = ∩ { M: M is a maximal ideal of a ring Q } .

∵ (Q , + , .) is a field ⟹ rad.(Q) = ({0} , + , .) .

Solution (4):- To find rad.(Z_{20}) ?

rad.$(Z_{20}) = \cap$ { M: M is a maximal ideal of a ring Z_{20} } .

$\because$ The maximal ideals in Z_{20} are : **20= 2(10)=4(5)**

1- $((\bar{2})\,,+_{20},\cdot_{20})$

2- $((\bar{5}),+_{20}\,,\cdot_{20})$

$\Longrightarrow$ rad.$(Z_{20}) = (\bar{2}) \cap (\bar{5}) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}, \ldots, \overline{18}\} \cap \{\bar{0}, \bar{5}, \overline{10}, \overline{15}\}$
$= \{\bar{0}, \overline{10}\} = (\overline{10})$.

Solution (5) :- To find rad.(Z_{60})

rad.$(Z_{60}) = \cap$ { M: M is a maximal ideal of a ring Z_{60} } .

$\because$ The maximal ideals in Z_{60} are: **60= 2.(30)=3.(20)=5.(12)= 6.(10)=4.(15)**

1- $((\bar{2})\,,+_{12},\cdot_{12})$

2- $((\bar{3}),+_{12}\,,\cdot_{12})$

3- $((\bar{5})\,,+\,,.)$

$\Longrightarrow$ rad.$(Z_{60})=(\bar{2})\cap(\bar{3})\cap(\bar{5})$

$\Longrightarrow$rad.$(Z_{60})=\{\bar{0}, \bar{2}, \bar{4}, \ldots\ldots, \overline{58}\}\cap\{\bar{0}, \bar{3}, \bar{6}, \ldots\ldots, \overline{57}\}\cap\{\bar{0}, \bar{5}, \overline{10}, \ldots, \overline{55}\}$

$\Longrightarrow$ rad.$(Z_{60})=\{\bar{0}, \overline{30}\}$

النقاط (6), (7), (8), (9) **Exc.**

Definition: - (**Semi Simple Ring**) حلقه شبه بسيطة

A ring (R, +,) is called **semi simple ring** if rad. (R) = (0) .

Example:-

The following rings (Q ,+ , .) , ($\mathcal{R}$, + , .) , (₵ , + , .) and (Z_p ,$+_p$, $\cdot_p$) where p is a prime number are semi simple ring.

(Q) Is(Z_{15} , $+_{15}$, $\cdot_{15}$) semi simple ring ?

Solution :-

∵ rad.(Z_{15}) = ∩ { M: M is a maximal ideal of a ring Z_{15} } .

Then ., the maximal ideals in Z_{15} are : **15=3(5)**

1- (($\bar{3}$) ,$+_{15}$, $\cdot_{15}$)

2- (($\bar{5}$),$+_{15}$, $\cdot_{15}$)

∴ rad.(Z_{15})=($\bar{3}$)∩ ($\bar{5}$)={ $\bar{0}$, $\bar{3}$, $\bar{6}$, $\bar{9}$, $\overline{12}$ } ∩ { $\bar{0}$,$\bar{5}$, $\overline{10}$ }={$\bar{0}$} .

Therefore , a ring (Z_{15} , $+_{15}$, $\cdot_{15}$) is semi simple ring

Theorem (2-23):-

If (I , + , .) is an ideal of a commutative ring with identity (R , + , .) . Then , I ⊆ rad.(R) iff each element of coset 1+I is an invertible .

Theorem (2-24):- In any ring (R , + , .) , a ∈rad.(R) iff 1- ra is an invertible for all r∈R.

Corollary :-

An element a is an invertible element in (R , +, .) iff the coset a+rad.(R) is an invertible element in ($R/rad.R$,+ , .) .

Theorem (2-25):- For any commutative ring with identity (R , + , .) Then, quotient ring ($R/rad.(R)$,+ , .) is semi simple ring .

Theorem (2-22):-

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