Definition :- (**The Image of Ring Homomorphism**)

Let f: (R , +, .) ⟶ (R` , +` , .`) be a ring homo . Then **the image of f** is denoted by **Im.f** or **f(R)** and is defined by :

Im .(f) = { à ∈ Ŕ : ∃ a ∈ R , s . t f(a)= à } .

بالتالي ممكن تعريف صورة دالة التشاكل (Im.f) بالشكل التالي ويستخدم التعريف ادناه بالامثلة اذا كان المطلوب في السؤال هو استخراج **Im.f**

So , Im.f = f(R) = { f(a) : ∀ a ∈ R }

Example :-

Let f:(Z ,+ , .) ⟶ ($\mathcal{R}$, +, .) be a function such that $f(a) = a$, $\forall\, a \in Z$
Find the **image of f**

Solution :-

< First to show , f is ring . homo>?

$\forall\, a$, b ∈ Z ⟹ $f\,(a) = a$ and $f\,(b) = b$, then

1- $f(a + b) = a + b = f\,(a) + f(b)$.

2- $f(a\,.\,b) = a\,.\,b = f\,(a)\,.\,f\,(b)$

⟹ **f is ring . homo**

⟹Im.f = f(Z) = { f(a) : ∀ a ∈ Z }

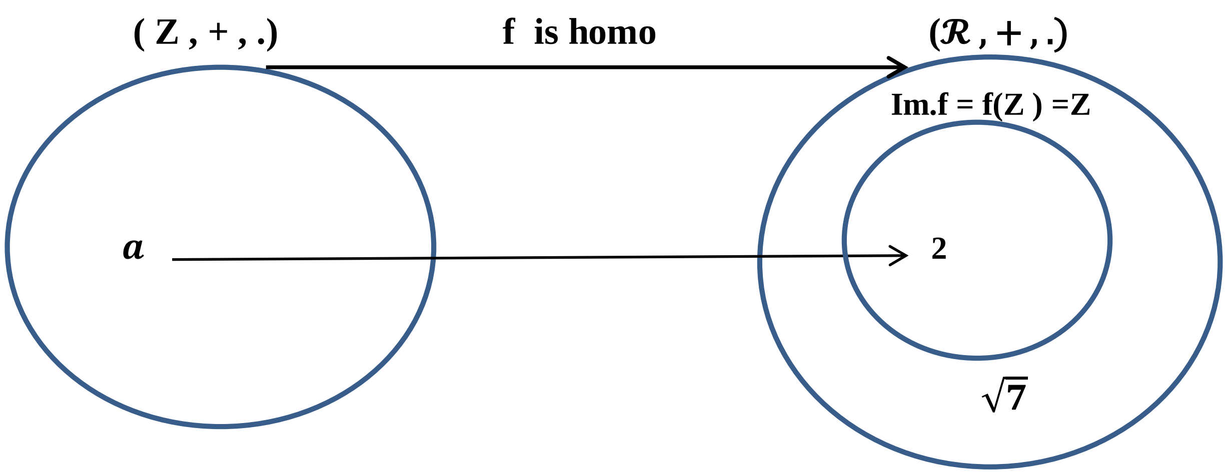
= { a : ∀ a ∈ Z } = Z .

ملاحظة : من المثال اعلاه نستنتج ان صورة دالة التشاكل Im.f هو حلقة جزئية (subring) ولكن ليس من الضروري ان يكون مثالي (ideal)

Im. (f)=Z is a **sub ring** of a ring $\mathcal{R}$,

But , Z **is not ideal** of a ring $\mathcal{R}$.

Since , $\exists\ 2 \in Z$, and $\exists \sqrt{7} \in \mathcal{R}$, but $2 . \sqrt{7} = 2\sqrt{7} \notin Z$.



التالي بعض أنواع التشاكل

Definition :- Let f: (R , +, .) ⟶ (R\`, +\`, .\`) be a function . Then

(1) f is called **Monomorphism** iff f is one to one and homo .

(2) f is called **Epimorphism** iff f is onto and homo.

(3) f is called **Isomorphism** iff f is one to one , onto and homo .

(4) f is called **Endomorphism** iff f is homomorphism and R=Ȑ .

(5) f is called **Automorphism** iff f is isomorphism and R=Ȑ .

Definition: - (**Isomorphic Rings**) الحلقات المتماثلة

Two rings (R , +, .) and (R\`, +\` , .\` ) are said to be **isomorphic** if there exists f : R⟶ R\` such that f is an isomorphism , and is denoted by (R≅ R\`) .

Example(1):- Let f: (Z ,+ , .) ⟶ (Z , + , .) be a function , S.t $f(n) = n.1$, $\forall n \in Z$. Show that : (f is ring homo , Epimorphism , Monomorphism , Isomorphism , Endomorphism and Automorphism)

Solution:- **First to show f is ring homo**

Let $n, m \in Z \implies f(n) = n.1$ and $f(m) = m.1$, thus

1- $f(n+m) = (n+m).1$

$= (n.1) + (m.1) = f(n) + f(m)$

2- $f(n.m) = (n.m).1$

$= (n.m).1^2$

$= (n.m)\, 1.1$

$= (n.1).(m.1) = f(n).f(m)$

∴ **f is ring homomorphism** ⋯ ⋯ (**i**)

< To show f is Epimorphism >? → **f is ring homo** ; → **f is on to**

by step(i) , f is ring homo

< Now, to show f is on to>?

$\because f(Z) = \{ f(n) , \forall n \in Z \}$

$= \{ n.1 , \forall n \in Z \}$

$= \{ n , \forall n \in Z \} = Z$

$\implies$ **f is on to** ⋯ ⋯ (**ii**)

$\implies$ **f is Epimorphism** .

Now ,to show , f is Monomorphism → f is ring homo
→ f is on to one

by step(i) , f is ring homo

< To show f is on to one >?

let $n, m \in Z$ such that $f(n) = f(m)$ < T.P. n = m >?

$\because f(n) = f(m)$

$\therefore \quad n.1 = m.1$

$\Longrightarrow n = m$,

$\Longrightarrow$ f is one to one $\cdots\cdots$ (**iii**) $\Longrightarrow$ **f is Monomorphism .**

< To show f is Isomorphism >?

by step (i) , (ii) and (iii) we get :

f is ring homo , onto and one to one $\Longrightarrow$ f is Isomorphism $\cdots\cdots$ (v)

< To show f is Endomorphism >?

Since, R= Z and R̀=Z

$\Longrightarrow$ R= R̀ $\cdots\cdots$ (**iv**)

Then, we get by step(i)[f is ring homo] and by step (iv) [R= R̀]

$\Longrightarrow$ f is Endomorphism .

< To show f is Automorphism >?

By step(v)[f is isomorphism] and step(iv) [R= R̀], then

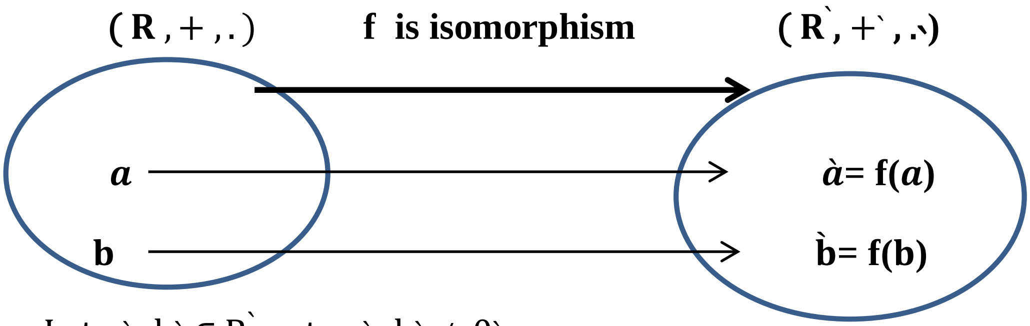
$\Longrightarrow$ f is Automorphism .

Exc. (2):- Let f: $(\mathcal{R}, +, .) \longrightarrow (\mathcal{R}, +, .)$ be a function such that $f(a) = 0, \forall\, a \in \mathcal{R}$ Is f Endomorphism ? واجب

Theorem (3.9):- Every isomorphic image of a ring without zero divisors is a ring without zero divisors .

Proof: Let $f:(R,+,.) \longrightarrow (R`, +`, .`)$ be an isomorphism function and $(R,+,.)$ is a ring without zero divisors

<T.P. (R`, +` , .`) is a ring without zero divisors>?



Let $a`, b` \in R`$ s. t . $a`, b` \neq 0`$

$\because$ f is 1-1 and on to $\Longrightarrow \exists!\ a, b \in R$ **s.t.** $\boxed{a` = f(a)}$ and $\boxed{b` = f(b)}$

Since, R without zero divisors

$\Longrightarrow a \cdot b \neq 0$

$\Longrightarrow f(a \cdot b) \neq f(0)$

$\Longrightarrow f(a) \cdot` f(b) \neq 0`$ (since f is homo . by Th (3-1) $f(0) = 0`$)

$\Longrightarrow a` \cdot` b` \neq 0`$

Therefore, $(R`, +`, .`)$ is a ring without zero divisors . □

Theorem (3-10):- Every isomorphic image of field is field . واجب