Infinite Series:

Definition: 1. Let $<a_n>$ be a sequence of real numbers and S_n is defined by

$$S_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^{n} a_k.$$

Then the sequence $<S_n>$ is called an infinite series. and denoted by $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$.

2. The number $S_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^{n} a_k$ is called n-th partial sum of the series.

3. The number a_n is called n-th term of the series.

4. The series $\sum a_n$ is said to be converges to S

iff $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = S$

(i.e iff $<S_n>$ converges to S). and we call the sum of the series and we write $\sum_{n=1}^{\infty} a_n = S$

or simply $a_1 + a_2 + \dots + a_n + \dots = S$

5. if $<S_n>$ is diverges sequence, then $\sum_{n=1}^{\infty} a_n$ is diverge i.e (if $\lim_{n \to \infty} S_n$ is not exist, then the series has no sum).

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Examples:-

1. let $<n> = <1, 2, 3, 4, \ldots, n, \ldots>$

then $S_1 = 1$

$S_2 = 1+2 = 3$

$S_3 = 1+2+3 = 6$

$\vdots$

$S_n = 1+2+3+\ldots+n$

2. let $<4> = <4, 4, \ldots, 4, \ldots>$

$S_1 = 4$

$S_2 = S_1 + 4 = 8$

$S_3 = S_2 + 4 = 12$

$\vdots$

$S_n = \underbrace{4+4+\ldots+4}_{n\text{-times}} = 4 \cdot n = 4n$

Find $S_{10} = \underbrace{4+4+\ldots+4}_{10\text{-times}} = (4)(10) = 40$

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Def:- A series of the form $a + ar + ar^2 + \dots + ar^{n-1} + \dots$ is called a geometric series.

Geometric Series theorem:-

IF $|r| < 1$, the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ where $a \neq 0$ converges to $\frac{a}{1-r}$

2. IF $|r| \geq 1$, the series diverges unless $a = 0$.
3. IF $a = 0$, the series converges to zero.

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Ex1. Find the geometric series with $a=4$, $r=\frac{1}{2}$

sol: The series is $4+4(\frac{1}{2})+4(\frac{1}{2})^2+4(\frac{1}{2})^3+\cdots$

$=4+2+1+\frac{1}{2}+\cdots$

The sum is $\frac{a}{1-r}=\frac{4}{1-\frac{1}{2}}=\frac{4}{\frac{1}{2}}=8$

Ex2 :- Find the geometric Series with $a=\frac{1}{3}$, $r=\frac{1}{4}$.

sol, The series is $\frac{1}{3}+\frac{1}{3}(\frac{1}{4})+\frac{1}{3}(\frac{1}{4})^2+\frac{1}{3}(\frac{1}{4})^3+\cdots$

The sum is $\frac{a}{1-r}=\frac{\frac{1}{3}}{1-\frac{1}{4}}=\frac{\frac{1}{3}}{\frac{3}{4}}=\frac{4}{9}$.

Ex3: Find $\sum_{n=1}^{\infty}\frac{1}{6^{n+1}}$

sol: $\sum_{n=1}^{\infty}\frac{1}{6^{n+1}}=\frac{1}{6^2}+\frac{1}{6^3}+\frac{1}{6^4}+\cdots$

$$\left.\begin{array}{l}\frac{1}{6^3}*6^2=\frac{1}{6}\\[2em]\frac{1}{6^4}*6^3=\frac{1}{6}\end{array}\right]=r$$

∴ The series is Geometric with

$a=\frac{1}{6^2}$, $r=\frac{1}{6}$, ∵ $-1<r<1$

∴ The series is converges to $\frac{a}{1-r}$

∴ The sum is $\frac{a}{1-r}=\frac{\frac{1}{36}}{1-\frac{1}{6}}=\frac{\frac{1}{36}}{\frac{5}{6}}=\frac{1}{30}$

Ex4: Find 1. $\sum_{n=0}^{\infty}\frac{3}{5^n}$ 2. $\sum_{n=1}^{\infty}\frac{9}{5^{n+1}}$ 3. $\sum_{n=1}^{\infty}\frac{2^n}{3^n}$

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Definition: The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called harmonic series and it is diverges

Example:- show that $\sum \frac{1}{n}$ is diverges.

sol/ let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$$S_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$

$$\therefore S_{2n} - S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$

$$n+1 \leq n+n$$

$$\Rightarrow \frac{1}{n+1} \geq \frac{1}{n+n}$$

$$\therefore \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \geq \underbrace{\frac{1}{n+n} + \cdots + \frac{1}{n+n}}_{n\text{-times}}$$

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \geq \frac{n}{2n} = \frac{1}{2}$$

$\therefore |S_{2n} - S_n| \geq \frac{1}{2}$, let $2n = m \Rightarrow |S_m - S_n| \geq \frac{1}{2}$

$\therefore \langle S_n \rangle$ is not Caushy sequence.

$\therefore \langle S_n \rangle$ diverges.

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Theorem $\sum a_n$ converges iff for any $\epsilon > 0$, there is an integer $N \ni \left| \sum_{i=n}^{m} a_i \right| < \epsilon$ if $m, n \geq N$.

proof :- since $\sum a_n$ converges

Then $\langle S_n \rangle$ is converges.

$\Rightarrow \langle S_n \rangle$ is caushy sequnce.

we get $\exists$ positive integer $N \ni |S_n - S| < \frac{\epsilon}{2}$ $\forall n \geq N$.

if $m \geq N$, $n \geq N$.

Then $|S_n - S| < \frac{\epsilon}{2}$ and $|S_m - S| < \frac{\epsilon}{2}$

$$\therefore |S_m - S_n| = |S_m - S + S - S_n|$$
$$\leq |S_m - S| + |S_n - S|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore \left| \sum_{i=1}^{m} a_i - \sum_{i=1}^{n} a_i \right| < \epsilon$ if $m \geq n$.

$\therefore \left| \sum_{i=n}^{m} a_i \right| < \epsilon$.

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$$\leq |S_m - S| + |S_n - S|$$

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$$\therefore \left|\sum_{i=1}^{m} a_i - \sum_{i=1}^{n} a_i\right| < \epsilon \quad \text{if } m \geq n.$$

$$\therefore \left|\sum_{i=n}^{m} a_i\right| < \epsilon .$$

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Corollary :- if $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

or if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

proof :- since $\sum a_n$ converges. then

for any $\epsilon > 0$, $\exists$ positive integer $N \ni$

$$\left| \sum_{i=n}^{m} a_i \right| < \epsilon \quad , \forall\ m \geq N,\ n \geq N.$$

$$\Rightarrow \left| \sum_{i=1}^{n+1} a_i - \sum_{i=1}^{n} a_i \right| < \epsilon.$$

$$\Rightarrow \left| \sum_{i=n}^{n+1} a_i \right| < \epsilon.$$

since $|a_{n+1}| < \epsilon$

we get $\lim_{n\to\infty} a_n = 0$.

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Theorem ② IF $\sum_{n=1}^{\infty} a_n = a$ and $\sum_{n=1}^{\infty} b_n = b$ are both exist. Finite then a. $\sum_{n=1}^{\infty} (a_n + b_n) = a + b$

b. $\sum_{n=1}^{\infty} K a_n = K a$ (K is any number.

proof ⓐ Let $A_n = a_1 + a_2 + \cdots + a_n$

$B_n = b_1 + b_2 + \cdots + b_n$.

Then the partial sum of $\sum (a_n + b_n)$ are

$$S_n = (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n)$$

$$= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n)$$

$$= A_n + B_n$$

Since $\lim_{n\to\infty} A_n = a$, $\lim_{n\to\infty} B_n = b$

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$\therefore \lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} S_n = \lim_{n\to\infty}(A_n+B_n) = \lim$

$$\therefore \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = a+b.$$

proof (b) The partial sum of $\sum_{n=1}^{\infty} Ka_n$ are

$$S_n = Ka_1 + Ka_2 + \cdots + Ka_n = KA_n.$$

since $\lim_{n\to\infty} KA_n = Ka. \Rightarrow$ we get $\langle KA_n \rangle$ c

$\therefore \sum Ka_n$ converges.

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Theorem ③: IF $\sum a_n$ diverges and if c is any number di[ffer]
zero, then the series of multiples $\sum c a_n$ diverges

Remark: If $\sum a_n = a$, $\sum b_n = b$, then
$$\sum (a_n - b_n) = \sum (a_n + (-b_n)) = \sum (a_n + (-1) b_n) =$$
$$= \sum a_n - \sum b_n = a - b$$
Then the series $\sum (a_n - b_n)$ is called the difference of $\sum a_n$ [and]
while $\sum (a_n + b_n)$ is called the sum.

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Theorem ③: IF $\sum a_n$ diverges and if c is any number di[ffrent from]
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Then the Series $\sum (a_n - b_n)$ is called the difference of $\sum a_n$ [and $\sum b_n$]
while $\sum (a_n + b_n)$ is called the sum.

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Theorem

if $\sum a_n$ converges and $\sum b_n$ diverges.
Then $\sum (a_n+b_n)$ diverges.

proof:- suppose that $\sum (a_n+b_n)$ converges.
Then $\sum (a_n+b_n) - \sum a_n = \sum b_n$ is converge
C!

Then $\sum (a_n+b_n)$ is diverges.

Definition:-

Let $\sum a_n$ and $\sum b_m$ be a Series in $\mathbb{R}$
then the product of them defined by

$$\sum a_n \cdot \sum b_m = (a_1+a_2+\cdots)(b_1+b_2+\cdots)$$
$$= (a_1 b_1 + a_1 \cdot b_2 + a_1 b_3 + \cdots + a_2 b_1 + a_2 b_2 + \cdots + a_3 b_1 + a_3 b_2 + \cdots$$

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Convergence Tests

1. The nth-term test

IF $\lim_{n\to\infty} a_n \neq 0$ or Limit is not exists, then $\sum a_n$ is diverges.

Ex 1 : $\sum n$ is diverges

since $\lim_{n\to\infty} n = \infty$ is not exists

Ex 2 : $\sum \frac{12n+20}{n}$ is diverges.

since $\lim_{n\to\infty} \frac{12n+20}{n} = 12 \neq 0$.

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2. The Ratio test

Let $\sum a_n$ be a Series with positive terms and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \alpha$.
then,

a. The Series converges if $\alpha < 1$.

b. " diverges if $\alpha > 1$.

c. The test is fails or (the test given no-information) if $\alpha = 1$.
(Try another test).

Ex1) Test the following Series for convergence or divergence. (using the ratio test).

1. $\sum \frac{1}{n!}$

$$a_n = \frac{1}{n!}, \quad a_{n+1} = \frac{1}{(n+1)!}$$

$$\therefore \alpha = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{1}{(n+1)!} * \frac{n!}{1}$$

$$= \lim_{n\to\infty} \frac{1}{(n+1)\cancel{n!}} * \frac{\cancel{n!}}{1} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

$\therefore$ The series is converges.

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4. $\sum \frac{1}{n^2}$

$$\alpha = \lim_{n \to \infty} \frac{1}{(n+1)^2} * \frac{n^2}{1} = \lim_{n \to \infty} \frac{n^2}{n^2+2n+1} = 1.$$

$\therefore$ The test is Fails.

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1. The nth Root test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq n_0$ and suppose that.

$$\alpha = \lim_{n\to\infty} \sqrt[n]{a_n}$$

Then: a. The series converges if $\alpha < 1$.

b. " " diverges if $\alpha > 1$.

c. The test is fails if $\alpha = 1$ (Try another test).

Ex 1: Determine whether the series is converges or diverges (using nth root test).

1. $\sum \frac{5^n}{n^3}$

$$\alpha = \lim_{n\to\infty} \sqrt[n]{\frac{5^n}{n^3}} = \lim_{n\to\infty} \frac{\sqrt[n]{5^n}}{\sqrt[n]{n^3}} = \lim_{n\to\infty} \frac{5}{(\sqrt[n]{n})^3} = 5 > 1.$$

∴ the series is diverges.

2. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

$$\alpha = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n^n}} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1.$$

∴ The series is converges.

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Def:- A series of the Form $\sum \frac{1}{n^p}$ is called p-series.

Remark: P-Series is converges if $P>1$

P-series is diverges if $P \leq 1$.

Ex:-

1. $\sum \frac{1}{n^3}$, $\sum \frac{1}{n^2}$, $\sum \frac{1}{n^{3/2}}$, $\sum n^{-7}$, $\sum \sqrt{\frac{1}{n^5}}$

All these series are converges.

2. $\sum \sqrt{n}$, $\sum \frac{1}{n}$, $\sum n^3$, $\sum n^5$, $\sum \frac{1}{n^{-3}}$

All these Series are diverges.

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1. Comparison Test for Series of non-negetive terms:-

Let $\sum a_n$ be a Series that has no negetive terms:-

a. Test for convergence of $\sum a_n$:-

The Series $\sum a_n$ converges if there is a convergent Series of non-negetive terms $\sum C_n$ with $a_n \le C_n$ for all $n > n_0$

b. Test for divergence of $\sum a_n$:-

The Series $\sum a_n$ diverges if there is a divergent series of non-negetive terms $\sum d_n$ with $a_n > d_n$ for all $n > n_0$.

Ex1. Test the Following Series 1. $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+5}$

Sol/ since $n^2+5 > n^2$

$$\therefore \frac{1}{n^2+5} < \frac{1}{n^2}$$

$\sum \frac{1}{n^2+5} < \sum \frac{1}{n^2}$ and since $\sum \frac{1}{n^2}$ is converges

($\sum \frac{1}{n^2}$ is p-Series, $p=2>1$)

$\therefore$ by comparison test

$\therefore \sum \frac{1}{n^2+5}$ is converges.

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Remark: P-Series is converges if $p>1$

P-series is diverges if $p \leq 1$.

Ex:-

1. $\sum \frac{1}{n^3}$, $\sum \frac{1}{n^2}$, $\sum \frac{1}{n^{3/2}}$, $\sum n^{-7}$, $\sum \sqrt{\frac{1}{n^5}}$

All these series are converges.

2. $\sum \sqrt{n}$, $\sum \frac{1}{n}$, $\sum n^3$, $\sum n^5$, $\sum \frac{1}{n^{-3}}$

All these series are diverges.

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3. $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

since $\ln(n) < n$

$\Rightarrow \frac{1}{\ln(n)} > \frac{1}{n}$

$\Rightarrow \sum \frac{1}{\ln(n)} > \sum \frac{1}{n}$ $(a_n > d_n)$

since $\sum \frac{1}{n}$ is diverges ($\sum \frac{1}{n}$ is harmonic Seri

$\therefore$ by comparsion test

$\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ is diverges.

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Theorem :- if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum a_n$ converges

proof :- let $A_n = \sum_{n=1}^{\infty} a_n$, $B_n = \sum_{n=1}^{\infty} |a_n|$

since $\sum_{n=1}^{\infty} |a_n|$ converges.

then $\langle B_n \rangle$ converges

$\Rightarrow$ $\langle B_n \rangle$ is Caushy sequence.

That mean $\forall \epsilon > 0$ be given , $\exists$ positive integer N

$\ni |B_m - B_n| < \epsilon$, $\forall n \geq N$, $m \geq N$.

$\Rightarrow \left| \sum_{i=1}^{m} |a_i| - \sum_{i=1}^{n} |a_i| \right| < \epsilon$

$\Rightarrow \sum_{i=n+1}^{m} |a_i| < \epsilon$

since $\left| \sum_{i=n+1}^{m} a_i \right| \leq \sum_{i=n+1}^{m} |a_i| < \epsilon$

$\therefore \left| \sum_{i=n+1}^{m} a_i \right| < \epsilon$.

$\therefore |A_m - A_n| < \epsilon$

$\Rightarrow$ $\langle A_n \rangle$ is Caushy sequence.

$\Rightarrow$ $\langle A_n \rangle$ is converges.

$\therefore \sum_{n=1}^{\infty} a_n$ converges.

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