Complex Algebra

Prof.Dr. Hayfa G. Rashid Dec.2020

المحاضرة الرابعة :

1. الاعداد العقدية بالنظام القطبي .. أمثلة تطبيقية

***Polar* Form or *Trigonometric* Form of a Complex Number**

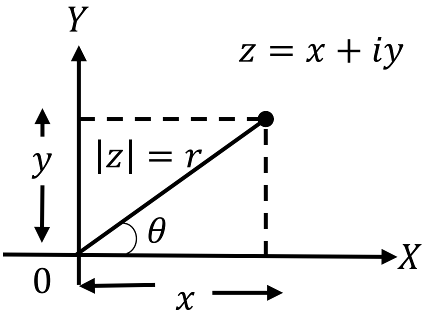
A complex number z, when written as

$$z = x + iy \quad or, \qquad z = x + jy$$

is said to be expressed in *rectangular form* ,also known as Cartesian coordinates .

But is also may be expressed in *polar form*

$$z = r\angle\,\theta$$



Geometrically, $|z|$ is the distance of the point from the origin r or can be represent by (A) .The number r (or A) is the length (or amplitude, modulus of the vector representing z; that is,

$$\boxed{r = |z|} = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} \quad (r \geq 0) \;\;, \;\; \theta = tan^{-1}(\frac{y}{x})$$

Conjugate of a Complex Number

The conjugate z^* or $\bar{z}$ of a complex number z is obtained by ***changing the sign of the imaginary part of the number.***

Conjugate of $z = z^* = Re\{z\} - iIm\{z\} = r\angle - \theta$

Now mathematical operation of complex number :

1. Addition $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
2. Subtraction $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$
3. Multiplication $z_1 z_2 = (r_1 r_2)\angle(\theta_1 + \theta_2)$
4. Division $\frac{z_1}{z_2} = (\frac{r_1}{r_2})\angle\,(\theta_1 - \theta_2)$

5. Reciprocal $\frac{1}{z} = \frac{1}{r}\angle -\theta$
6. Square root $\sqrt{z} = \sqrt{r}\angle\ {}^{\theta}/_{2}$
7. Complex conjugate $z^* = x - jy = r\angle -\theta$
   $0^o = \mp 360^o = +1 = 1\angle\ 0^o = 1 + j0$

   $+90^o = +\sqrt{-1} = +j = 1\angle +90^o = 0 + j1$

   $-90^o = -\sqrt{-1} = -j = 1\angle -90^o = 0 - j1$

   $\mp 180^o = (\ \sqrt{-1}\ )^2 = -1 = 1\angle \mp 180^o = -1 + j\ 0$

Converting Polar Form into Rectangular Form, (P→R)

(From **P**olar form to **R**ectangular form)

$$6\angle 30^o = x + jy$$

However,

$$x = A\cos\theta \quad , y = A\sin\theta$$

Therefore,

$$6\angle 30^o = (6\cos\theta) + j(6\sin\theta)$$
$$=(6\cos\theta) + j(6\sin\theta)$$
$$=(6\cos 30^o) + j(6\sin 30^o)$$
$$=(6 \times 0.866) + j(6 \times 0.5)$$
$$= 5.2 + j3$$

Converting Rectangular Form into Polar Form, (R→P)

(From **R**ectangular form to **P**olar form)

$$(\mathbf{5.2 + j3}) = \boldsymbol{A\angle\theta}$$

Where $\boldsymbol{A = \sqrt{5.2^2 + 3^2} = 6}$ and $\boldsymbol{\theta = \tan^{-1}\frac{3}{5.2} = 30^o}$

Hence , $(\mathbf{5.2 + j3}) = \mathbf{6\angle 30^o}$

Multiplication in Polar Form

$$Z_1Z_2 = (A_1A_2)\angle(\theta_1 + \theta_2)$$

Multiplying together $6\angle 30^\circ$ and $8\angle -45^\circ$ in polar form gives us:

$$Z_1Z_2 = (6 \times 8)\angle(30^o + (-45^o)) = 48\angle -15^o$$

Division in Polar Form

$$\frac{Z_1}{Z_2} = \left(\frac{A_1}{A_2}\right)\angle\theta_1 - \theta_2$$

$$\frac{Z_1}{Z_2} = \left(\frac{6}{8}\right)\angle 30^o - (-45^o) = 0.75\angle 75^o$$

Polar Coordinates

Let r and θ be polar coordinates of the point (x, y) corresponding to a nonzero complex number $z = x + iy$

Since, $x = r\cos\theta$, $y = r\sin\theta$

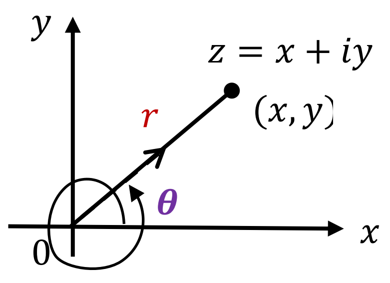
$$\boxed{z = r(\cos\theta + i\sin\theta)} \quad , \qquad \textit{polar form of number } z$$

For example,

$$1 + i = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = \sqrt{2}\left[\cos\left(\frac{-7\pi}{4}\right) + i\sin\left(\frac{-7\pi}{4}\right)\right]$$

The number θ is called an *argument* of z, (also called, '*phase*' *and* '*angle*')

θ is the directed angle measured from the positive x-axis to a point p and called (*argument* of z) and is denoted by : arg ; $\theta = \arg z_1$



Hence the angles will be measured in *radian* and positioned in the *counterclockwise sense*. $\boxed{\tan\theta = \frac{y}{x}}$

Important identity

1. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
2. $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$

Question : Find the value of (iz)

$$iz = (1)\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] r(\cos\theta + i\sin\theta)$$

i z

$$iz = r\left[\cos\left(\theta + \frac{\pi}{2}\right) + i\sin\left(\theta + \frac{\pi}{2}\right)\right]$$

iz has an important applications in Optics i.e. (*circular polarization*).

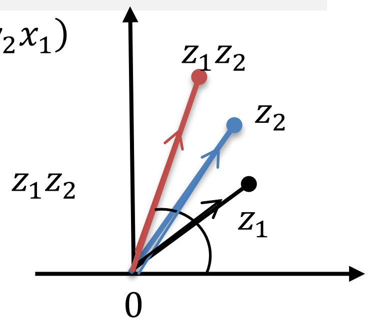
Example:

$\theta_1 = \arg(1+i)$, $\theta_1 = arg(-1-i) \Longrightarrow \tan\theta_1 = \tan\theta_2 = 1$….**PLOT!!**

Question : Use Cartesian coordinate system representation of z , simplified the multiplication of two different complex numbers.

$$z_1 z_2 = (x_1 + iy_2)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + y_2x_1)$$

$$(\mathbf{1} + \boldsymbol{i})(\mathbf{2} + \mathbf{3}\boldsymbol{i}) = \mathbf{2} + \mathbf{5}\boldsymbol{i} - \mathbf{3} = -\mathbf{1} + \mathbf{5}\boldsymbol{i}$$

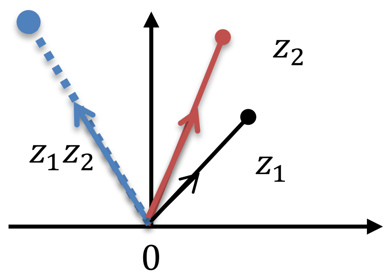


$z_1 = (1+i) \rightarrow x_1 = 1, y_1 = 1,\ r_1 = \sqrt{2},\ \theta_1 = 45°$,

$z_2 = (2+3i)$, $x_2 = 2, y_2 = 3,\ r_2 = \sqrt{13}$

Then $\theta_2 = \tan^{-1}\frac{3}{2}$ or $\theta_2 = 56.31°$,

For the ***final results*** $-1+5i$, will be :



$r = 5.1$, $\theta = \tan^{-1} -\frac{5}{1}$, $\theta = 101.31°$ دقق الاجابة