# Complex Functions

المحاضرة السابعة :

## الدوال العقدية مع أمثلة تطبيقية

## Complex Functions الدوال العقدية

Real part of f(z) — Imaginary part of f(z)

$$w = f(z) = U(x, y) + iV(x, y)$$

The variable $z$ is sometimes called an *independent* variable, $w$ is called a *dependent* variable.

**Example:**

If $f(z) = z^2$, then find $f(z)$ if $z = 2i$

$$f(2i) = (2i)^2 = -4$$

**Example**

If $w = z^2$, then $u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$, and the transformation is

$$f(z) = z^2 = (x + iy)^2$$
$$f(z) = x^2 + 2xyi - y^2$$

$$U = x^2 - y^2, \qquad V = 2xy$$

**Example:**

Illustrate the function

$$f(z) = \underbrace{\sqrt{x^2 + y^2}}_{u(x,y)} + i\underbrace{(-y)}_{v(x,y)}$$

$$\Rightarrow u(x, y) = \sqrt{x^2 + y^2}$$
$$\Rightarrow v(x, y) = -y$$

## *Elementary Functions*

**1. Polynomial function are defined by**

$$w = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = P(z)$$

Where $a_0 \neq 0, a_1, \dots, a_n$ are complex constants and $n$ is a positive integer called the degree of the polynomial $P(z)$.

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*Complex Exponential function*

$$e \quad , i \quad , \pi$$

**Euler's formula**

$$e^{i\theta} = \cos\theta + i\sin\theta$$

**De'Moiver theorem**

$$e^{i(n\theta)} = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

**2. Rational Algebraic functions are defined by** $w = \frac{P(z)}{Q(z)}$

Where $P(z), Q(z)$ are polynomials $Q(z) \neq 0$.

**3. Exponential functions are defined by**

$$w = e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$

Where $e = 2.71828 \ldots$ is the natural base of logarithms .If $a$ is real and positive, we define: $a^z = e^{z \ln a}$ ( ***prove***)

*Where* $\ln a$ *is the natural logarithm of* $a$.

مهم

***Proof***

$$w = a^z \Longrightarrow \ln w = \ln a^z \Longrightarrow \ln w = z \ln a$$

$$\therefore w = e^{z \ln a}$$

* دالة اسية مرفوعة الى ( الاس مضروب في لوغاريتم الاساس )

Complex exponential functions have properties similar to those of real exponential function. For example,

$$e^{z_1} * e^{z_2} = e^{z_1+z_2},$$

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$$

**4. Trigonometric functions** (or circular functions) $\cos z, \sin z$, etc. can be defined in terms of exponential functions as follows.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

$$\cot z = \frac{\cos z}{\sin z} = \frac{i\left(e^{iz} + e^{-iz}\right)}{e^{iz} - e^{-iz}}$$

- $\sin^2 z + \cos^2 z = 1, \quad 1 + \tan^2 z = \sec^2 z, \quad 1 + \cot^2 z = \csc^2 z$
- $\sin(-z) = -\sin z, \quad \cos(-z) = \cos z, \quad \tan(-z) = -\tan z$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$
$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

## 5. Hyperbolic Functions are defined as follows

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^{z} + e^{-z}}, \qquad \operatorname{csch} z = \frac{1}{\sinh z} = \frac{2}{e^{z} - e^{-z}}$$

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}, \qquad \operatorname{csch} z = \frac{\cosh z}{\sinh z} = \frac{e^{z} + e^{-z}}{e^{z} - e^{-z}}$$

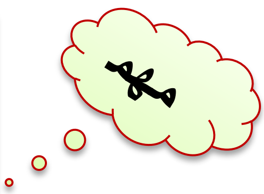
The following properties hold ( تعطى )

- $\cosh^2 z - \sinh^2 z = 1, \quad 1 - \tan^2 z = \operatorname{sech}^2 z, \quad \coth^2 z - 1 = \operatorname{csch}^2 z$
- $\sinh(-z) = -\sinh z, \quad \cosh(-z) = \cosh z, \quad \tanh(-z) = -\tanh z$
- $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$

- The following relations exist between the trigonometric or circular functions and the hyperbolic functions.

$$\sin iz = i \sinh z, \quad \cos iz = \cosh z, \quad \tan iz = i \tanh z$$
$$\sinh iz = i \sin z, \quad \cosh iz = \cos z, \quad \tanh iz = i \tan z$$



## Logarithmic Functions:

If $z = e^w$, then we write $w = \ln z$, called the natural logarithm of $z$

$$w = \ln z, \qquad z = re^{i\theta}$$

$$w = \ln(re^{i\theta}) = \ln r + \ln e^{i\theta}$$

$$\boxed{w = \ln z = \ln r + i(\theta + 2k\pi)}, \qquad k = 0, \pm 1, \pm 2, \dots$$

Where $z = re^{i\theta} = re^{i(\theta + 2k\pi)}$

$\ln z \Longrightarrow$ is multiple-valued function. ***The principle branch*** of principle value of $\ln z$ is defined as:

$$\ln r + i\theta, \qquad \text{where } 0 \le \theta \le 2\pi$$

- The logarithmic function can be defined for real bases other than $e$. If

$$\boxed{z = a^w}, \qquad w = \log_a z$$

Where $a > 0$ and $a \neq 0{,}1$.

In this case $\boxed{z = e^{w \ln a}}$ and so $\boxed{w = (\ln z) / \ln a)}$

## 6. Inverse Trigonometric Functions ( تعطى )

If $\boxed{z = \sin w}$, the $w = \sin^{-1} z$ is called the inverse sine of $z$: or arc sine of $z$.

$$w = \sin^{-1} z \Longrightarrow \text{(multiple-valued function)}$$

- $\sin^{-1} z = \frac{1}{i}\ln\left(iz + \sqrt{1 - z^2}\right)$, • $\cos^{-1} z = \frac{1}{i}\ln\left(z + \sqrt{z^2 - 1}\right)$

## 7. Inverse Hyperbolic functions( تعطى )

If $= \sinh w$, the $w = \sinh^{-1} z$ is called the inverse hyperbolic sine of $z$.

$$w = \sinh^{-1} z \Longrightarrow \text{(multiple-valued function)}$$

- $\sinh^{-1} z = \ln\left(z + \sqrt{z^2 + 1}\right)$, $\cosh^{-1} z = \ln\left(z + \sqrt{z^2 - 1}\right)$,

### *Example*

**1.** Prove that:

**a)** $e^{z_1} * e^{z_2} = e^{z_1 + z_2}$

**b)** $|e^z| = e^x$

**c)** $e^{z + 2k\pi i} = e^z$, $k = 0, \pm 1, \pm 2, \dots$

***Solution***

**a)** By definition

$$e^z = e^x(\cos y + i \sin y) \qquad \text{where} \begin{cases} z = x + iy \\ z_1 = x_1 + iy_1 \\ z_2 = x_2 + iy_2 \end{cases}$$

$$\therefore e^{z_1} * e^{z_2} = e^{x_1}(\cos y_1 + i\sin y_1) * e^{x_2}(\cos y_2 + i\sin y_2)$$
$$= e^{x_1} * e^{x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2)$$
$$= e^{x_1+x_2}\cos(y_1+y_2) \ + i\, e^{x_1+x_2} sin(y_1+y_2)$$

**b)** $|e^z| = |e^x(\cos y + i\sin y)| = |e^x||\cos y + i\sin y|$
$$= e^x * 1 = e^x$$

**c)** By part (a)

$$e^{z+2k\pi i} = e^z * e^{2k\pi i} = e^z(\cos 2k\pi + i\sin 2k\pi) = e^z$$

$2k\pi i \Longrightarrow$ is a ***period*** of the function ,In particular, $\boldsymbol{e^z}$ ***has*** ***period*** $\boldsymbol{2\pi i}$.

**2. Find the value of $\boldsymbol{ln(1-i)}$. What is the principle value**?

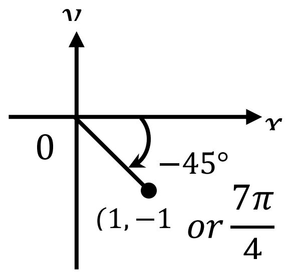
Or, determine the Principle value of $ln(1-i)$.

$$ln\, z = ln\left(re^{i\theta}\right) = ln\, r + i\theta$$

$z = 1 - i \Longrightarrow x = 1, y = -1,$

$$r = \sqrt{2}$$

$$\theta = -45°, \text{or } -\frac{\pi}{4}$$




$\ln z = \ln(1-i) = \ln\sqrt{2} + i\left(-\frac{\pi}{4} + 2k\pi\right) =$

$\frac{1}{2}ln\, 2 + \frac{7\pi}{4}i + 2k\pi i$

The ***principle value*** is $(\boldsymbol{k=0})$, $\left(\boldsymbol{\frac{1}{2}\ln 2 + \frac{7\pi}{4}i}\right)$


**a)** Find the principle value of $i^i$.

***Solution***

**(a)** by definition $z^i = e^{i\ln z} = e^{i\{\ln r + i(\theta+2k\pi)\}} = e^{i\ln r - (\theta+2k\pi)}$

$$= e^{-(\theta+2k\pi)}\{\cos(\ln r) + i\sin(\ln r)\}$$

$0 \le \theta \le 2\pi$

$$k = 0, \text{then } z^i = e^{-\theta}\{\cos(\ln r) + i\sin(\ln r)\}$$

**(b)** By definition, $i^i = e^{i\ln i}$, the principle value $= i^i = e^{-\pi/2}$

Now ,

Let $w = Z^Z$ , $ln\, w = ZlnZ$

**Example :** $(1+i)^{(2-5i)}$

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**Method#1**

**Let** $, w=(1+i)^{(2-5i)}$

$lnw=(2-5i)\,ln(1+i)\quad , \quad ln\,z=lnr+i\theta\,,$

$x=1\,, y=1\,, r=\sqrt{2}\,,\quad \theta=45^0\,, or\quad \frac{\pi}{4}$

$ln(1+i)=ln\sqrt{2}+i\frac{\pi}{4}$

$lnw=(2-5i)\,ln(1+i)=(2-5i)[ln\sqrt{2}+i\frac{\pi}{4}]$

Continue!!!

$lnw=2ln\sqrt{2}+i\frac{\pi}{2}-5ln\sqrt{2}\,i+\frac{5\pi}{4}$

Then taking the exponential of both sides ……

**Method#2**

$$w=(1+i)^{(2-5i)}$$

$$w=e^{(2-5i)ln(1+i)}=e^{(2-5i)[ln\sqrt{2}+i\frac{\pi}{4}]}=e^{2ln\sqrt{2}+i\frac{\pi}{2}-5ln\sqrt{2}\,i+5\frac{\pi}{4}}$$

$$w=e^{\left(2ln\sqrt{2}+5\frac{\pi}{4}\right)+i\left(\frac{\pi}{2}-5ln\sqrt{2}\right)}$$

Continue!!!

$$w=e^{\left(2ln\sqrt{2}+5\frac{\pi}{4}\right)}\times e^{i\left(\frac{\pi}{2}-5ln\sqrt{2}\right)}$$

Make use of Euler's formula

$$e^{i\theta}=cos\theta+isin\theta$$

**Solve :** $i^{(2i)}\,,(1-3i)^{(2.5-1.3i)}\,,\ 10^{-7i}\,,\ (-9+\sqrt{3}\,i)^{(-6i)}$

(a) $w=i^{2i}$

$lnw=(2i)ln(i)$

$$ln\,z=lnr+i\theta$$

$$ln\,i=ln1+i\frac{\pi}{2}=0+i\frac{\pi}{2}\,,\quad ln\,i=i\frac{\pi}{2}$$

Then ,

$$\ln w = (2i)\left[i\frac{\pi}{2}\right] = -\pi$$

$$\ln w = -\pi$$

*Taking* ***exponential*** *for both sides:*

$$e^{\ln w} = e^{-\pi} \rightarrow \quad w = e^{-\pi}$$

$$w = e^{-\pi}$$

*(b)* $(1-3i)^{(2.5-1.3i)}$

$$w = (1-3i)^{(2.5-1.3i)} \quad \rightarrow \ln w = \ln\left[(1-3i)^{(2.5-1.3i)}\right]$$

$$\ln w = (2.5-1.35i)\ln(1-3i)$$

$$\ln z = \ln r + i\theta$$

**$ln(1-3i)$ !!!!!!**

$x = 1$ , $\quad y = -3$ , $r = 3.16$, $\theta = -7.1.6^o$ or , $\theta = 288.4^o$ , $\theta = 1.6\pi$

$$\ln(1-3i) = \ln(3.16) + 288.4^o\, i$$

$$\therefore \quad \ln w = \ln(3.16) + 288.4^o\, i = 1.15 + 288.4^o\, i$$

Then ,

$$e^{\ln w} = e^{1.15+288.4^o\, i}$$

$$w = e^{1.15} \times e^{288.4^o\, i}$$

$$(1-3i)^{(2.5-1.3i)} = 3.16[\cos 288.4^o + i\sin 288.4^o]$$

$$(1-3i)^{(2.5-1.3i)} = (3.16 \times 0.316) - i(3.16 \times 0.949)$$

$$(1-3i)^{(2.5-1.3)} = 1.14 - 2.99i$$