Prof.Dr. Hayfa G. Rashid 2021

Analytic (Cauchy –Remann Conditions) and Harmonic complex functions (Laplace Equation)

الفصل الثاني

المحاضرة الثانيــة

✓ الدوال العقدية التحليلية و (شروط كوشي- ريمان)

✓ الدوال العقدية التوافقية (معادلة لابلاس)

Previous lecture ,

Complex function $\boldsymbol{w = f(z) = U(x,y) + i\,V(x,y)}$

- Complex function to be *analytic* must satify *Cauchy – Remann Conditions* , that is :

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

Example 1: Does the function $f(z) = e^z$ analytic ?.

Solution

$$f(z) = e^z = e^{x+iy} = e^x \times e^{iy}$$

$$e^z = e^x[cosy + isiny] = e^x cosy + i\, e^x\, siny$$

$$\therefore \quad u(x,y) = e^x cosy \quad , \quad v(x,y) = e^x\, siny$$

C-R-C's

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$$\frac{\partial U}{\partial x} = e^x cosy \quad , \quad \frac{\partial V}{\partial x} = e^x \, siny$$
$$\frac{\partial U}{\partial y} = -e^x siny \quad , \quad \frac{\partial V}{\partial y} \; e^x cosy$$

The conditons are ***satisfied*** the the function $f(z) = e^z$ is *analtyic*

- A complex function to be *harmonic* must satify *Laplace equation* , that is :

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \quad , or \quad U_{xx} + U_{yy} = 0$$
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad , or \quad V_{xx} + V_{yy} = 0$$

Or ,

$$\nabla^2 \psi = 0 \quad where \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The ' *operator* ' ∇^2 is called " *Laplacian*" .

Example 2 : Does the function $f(z) = e^z$ harmonic ?

Solution

$$\therefore \quad u(x, y) = e^x cosy \quad , v(x, y) = e^x \, siny$$

$$\frac{\partial U}{\partial x} = e^x cosy \quad , \quad \frac{\partial V}{\partial x} = e^x \, siny$$

$$\frac{\partial^2 U}{\partial x^2} = e^x cosy \quad , \quad \frac{\partial^2 V}{\partial x^2} = e^x \, siny$$
$$\frac{\partial U}{\partial y} = -e^x siny \quad , \quad \frac{\partial V}{\partial y} \; e^x cosy$$
$$\frac{\partial^2 U}{\partial y^2} = -e^x cosy \quad , \quad \frac{\partial^2 V}{\partial y^2} = -e^x \, siny$$

The conditons are ***satisfied*** the the function $f(z) = e^z$ is *harmonic function* .

Example 3 : Does the function $f(z) = cosz$ analytic , harmonic ?

Solution

$$f(z) = cosz = cosx\ coshy - i\ sinx\ coshy$$

$$\therefore\ u(x,y) = cosx\ coshy \quad , v(x,y) = -sinx\ coshy$$

$$\frac{\partial U}{\partial x} = -sinx\ coshy \ , \quad \frac{\partial V}{\partial x} = -cosx\ sinhy$$

$$\frac{\partial^2 U}{\partial x^2} = -cosx\ coshy \ , \quad \frac{\partial^2 V}{\partial x^2} = sinx\ sinhy$$

$$\frac{\partial U}{\partial y} = cosx\ sinhy \ , \quad \frac{\partial V}{\partial y} = -sinx\ coshy$$

$$\frac{\partial^2 U}{\partial y^2} = cosx\ coshy \ , \quad \frac{\partial^2 V}{\partial y^2} = -sinx\ sinhy$$

The conditons are ***satisfied*** the the function $f(z) = cosz$ is *analytic* and *harmonic function .*

Example 4 : Does the function $f(z) = lnz$ analytic , harmonic ?

Note : $\frac{d}{dz}tan^{-1}z = \frac{-1}{1+z^2}$

Solution

$$f(z) = lnz = ln(x + iy)$$

$$lnz = \frac{1}{2}ln(x^2 + y^2) + i\ tan^{-1}(y/x)$$

$$\therefore U(x,y) = \frac{1}{2}ln(x^2 + y^2)$$

$$\frac{\partial U}{\partial x} = \frac{1}{2} \times \frac{1}{x^2 + y^2} \times 2x = \frac{x}{x^2 + y^2}$$

$$\frac{\partial U}{\partial y} = \frac{1}{2} \times \frac{1}{x^2 + y^2} \times 2y = \frac{y}{x^2 + y^2}$$

$$V(x,y) = tan^{-1}(y/x)$$

$$\frac{\partial V}{\partial x} = \frac{1}{1+(y/x)^2} \times (-x^2 y) = \frac{x}{x^2+y^2}$$

$$\frac{\partial V}{\partial y} = \frac{1}{1+(y/x)^2} \times (-x^{-1})$$

Activity :

Does the functions

1. $f(z) = z + \bar{z}$ analytic , harmonic ?
2. $f(z) = \frac{1}{z}$ analytic , harmonic ?
3. $f(z) = \frac{z+i}{z-i}$ analytic , harmonic ?
4. $f(z) = zlnz$ analytic , harmonic ?
5. $f(z) = \boldsymbol{i}\, cosz$ analytic , harmonic?