Complex Power Series Expansion

الفصل الثاني

المحاضرة الثالثـــة

متسلسلات الدوال العقديـــة

- ✓ متسلسلة تايلــر (**Taylor Series**) ومتسلسلة ماكلورين (**Maclaurin Series**)
- ✓ متسلسلة لورنت (**Laurent Series**)

Taylor Series

Suppose that a function f is analytic throughout a circle $|z - z_0| < R_0$, centered at z_0 and with radius R_0 . Then $f(z)$ has the power series representation.

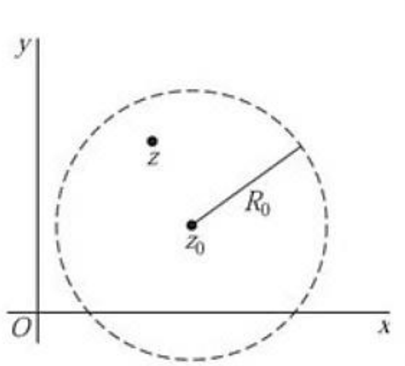
$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$$

Where ,

$$n! = n(n-1)(n-2)\ldots .3.2.1$$

$$0! = 1 \;,\; 1! = 1 \;,\; 2! = 2 * 1 = 2 \;,$$

$$3! = 3 * 2 * 1 = 6 , \ldots . etc$$



ملاحظـة : عنما تكون $z = 0$ فان متسلسلة **تايلر** **تؤول** الى متسلسلة **ماكلورين** .

Here some function expansion and convergent region:

1. $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad |z| < \infty$
2. $sinz = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots \qquad |z| < \infty$
3. $cosz = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots \qquad |z| < \infty$
4. $coshz = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots \qquad |z| < \infty$

5. $(1+z)^{\propto} = 1 + \propto z - \frac{\propto(\propto-1)z^2}{2!} + \cdots \qquad |z| < 1$

is the ***binomial theorem*** *for* $\propto = -1$ ***gives formula 8.***

6. $\ln z$ ***undefined*** *at* $z = 1$ ***Why!!!!***

7. $\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots\ldots \qquad |z| < 1$

8. $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots \qquad |z| < 1$

9. $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1$

The first five expansion are valid for all z, whilst the last three are only valid for $|z| < 1$.

Example 1 Expand $f(z) = \ln(1+z)$

Solution

$$f(z) = \ln(1+z) \quad , \quad f(0) = 0$$

$$f'(z) = \frac{1}{1+z} \quad , \quad f'(0) = 1$$

$$f''(z) = \frac{-1}{(1+z)^2} \quad , \quad f''(0) = -1$$

$$f^{(3)}(z) = \frac{(-1)(-2)}{(1+z)^2} \quad , \quad f^{(3)}(0) = 2$$

....

....

$$f^{(n+1)}(z) = \frac{(-1)^n \; n!}{(1+z)^{(n+1)}} \quad , \quad f^{(n+1)}(0) = (-1)^n \; n!$$

$$f(z) = \ln(1+z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots$$

$$f(z) = \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots.$$

Example 2 Expand $f(z) = \ln\frac{(1+z)}{(1-z)}$

Solution

$$f(z) = \ln\frac{(1+z)}{(1-z)} \quad ,$$

$$f(z) = \ln\frac{(1+z)}{(1-z)} = \ln(1+z) - \ln(1-z)$$

$$ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots.$$

$$ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} + \cdots.$$

$$ln(1+z) - ln(1-z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots. + z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots.$$

$$= 2z + \frac{2z^3}{3} + \cdots. = 2(z + \frac{2z^3}{3} + \frac{z^5}{5} \ldots.) = \sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}$$

$$ln\frac{(1+z)}{(1+z)} = \sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}$$

Example 3 Expand $f(z) = \frac{1}{(1+z)^2}$

Solution

$$f(z) = \frac{1}{(1+z)^2}$$

$$f(z) = \frac{1}{(1+z)^2} = (1+z)^{-2}\ , \quad f(0) = 1$$

$$f'(z) = \frac{-2}{(1+z)^3}\quad , \quad f'(0) = -2$$

$$f''(z) = \frac{6}{(1+z)^4}\quad , \quad f''(0) = 6$$

$$\therefore \quad \frac{1}{(1+z)^2} = 1 - 2z + 3z^2 + \cdots \qquad provided \quad |z| < 1$$

Example 4 Expand $f(z) = \frac{1}{1+z}$

Solution

$$f(z) = \frac{1}{1+z} = (1+z)^{-1}$$

$$f(z) = \frac{1}{1+z} \quad , f(0) = 1$$

$$f'(z) = \frac{-1}{(1+z)^2}\quad , \quad f'(0) = -1$$

$$f''(z) = \frac{2}{(1+z)^3}\quad , \quad f''(0) = 2$$

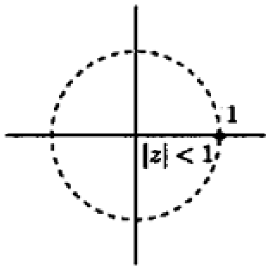
....

....

$$\therefore \quad \frac{1}{1+z} = 1 - z + z^2 + \cdots$$

Example 5 If $|z| < 1$, expand $f(z) = \frac{1}{1-z}$

Solution



$$f(z) = \frac{-1}{1-z} = (1+z)^{-1}$$

$$f(z) = \frac{1}{1-z} \quad , f(0) = 1$$

$$f'(z) = \frac{1}{(1-z)^2} \quad , \quad f'(0) = 1$$

$$f''(z) = \frac{2}{(1-z)^3} \quad , \quad f''(0) = 2$$

....

....

$$\therefore \quad \frac{1}{1-z} = 1 + z + z^2 + \cdots$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad , \quad |z| < 1$$

Example 6 Expand $f(z) = e^{i\theta}$

Solution

$$\because \quad e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Then

$$e^{i\theta} = 1 + (i\theta) + + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$ "*Euler formula*"

Activity : $e^{i\pi} + 1 = 0$

Is called the most beautiful equation in all of mathematics

- ✓ It is an *identify* that contains the most beautiful entities encountered in math, namely π, i, e, 0 and 1.
- ✓ It **combines** the real and the imaginary.

Prof.Dr. Hayfa G. Rashid 2021

In mathematics,***Euler's*** ***identity*** (also known as ***Euler's*** ***equation***) is the equality

$$e^{i\pi} + 1 = 0$$

where

e	is Euler's number, the base of natural logarithms,
i	is the imaginary unit, which by definition satisfies $i^2 = -1$, and
π	is pi, the ratio of the circumference of a circle to its diameter.

Euler's ***identity*** is a special case of *Euler's* ***formula***, which states that for any real number x,

$$e^{ix} = \cos x + i\sin x$$

where the inputs of the trigonometric functions sine and cosine are given in ***radians***. In particular, when $x = \pi$

$$e^{i\pi} = \cos\pi + i\sin\pi.$$

Since

$$\cos\pi = -1$$

and

$$\sin\pi = 0$$

it follows that $e^{i\pi} = -1 + 0i$

which *yields Euler's identity* : $e^{i\pi} + 1 = 0$

Activities

- $e^0 = 1$
- $e^z \neq 0$
- $|e^z| = e^x \equiv e^{Rez}$
- $arg(e^z) = y + 2n\pi \equiv Imz \ , (n = 0, \mp 1, \mp 2, \dots)$

We know that $z' = x' + iy'$, *then* $\arg(z') = tan^{-1}(\frac{y'}{x'})$

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \cos(y) + ie^x \sin(y) = x' + iy'$$

$$\arg(e^z) = \tan^{-1}\left(\frac{y'}{x'}\right) = \tan^{-1}\left(\frac{e^i \sin(y)}{e^x \cos(y)}\right) = \tan^{-1}(\tan(y)) = y + 2k\pi$$

$$\Rightarrow \quad arg(e^z) = y + 2\pi k$$

- $\overline{e^z} = e^{\bar{z}}$
- $\ln(e^z) = z + 2\pi i \quad ,(n = 0, \mp 1, \mp 2, \dots)$
- $e^{\log z} = z$
- $\ln(z^{1/k}) = \frac{1}{k}\ln z \quad , \quad (k = \mp 1, \mp 2, \dots)$

Homework

1. Expand the follwing function

(a) $e^{-z} \quad ; \quad z = 0$

(b) $\cos z \quad ; \quad z = \pi/2$

(c) $\frac{1}{1+z} \quad ; \quad z = 1$

(d) $\ln z \quad ; \quad z = 2$

(e) $z\, e^{2z} \quad ; \quad z = -1$

2.Show that : $\quad \sin z = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \cdots \quad , \; |z| < \infty$

3.Show that : $\quad \tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots \quad , \; |z| < 1$

4.Show that : $\quad \sec z = 1 + \frac{z^2}{2} + \frac{5z^4}{24} + \cdots \quad , \; |z| < \pi/2$

5.Show that : $\quad \csc z = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots \quad , \; 0 < |z| < \pi$

6. Expand : $\quad \tan^{-1}(iz) = \quad$ ... ...

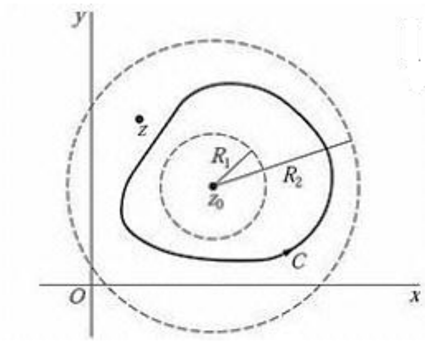
Laurent's Series

The method of *Laurent series expansions* is an important tool in complex analysis. Where a *Taylor series* can only be used to describe the *analytic part* of a function, Laurent series allows us to *work around the singularities* of a complex function. To ***do this***, we need to determine ***the singularities*** of the function and can then ***construct several concentric rings*** *with the same center* z_0 based on those singularities and obtain a unique Laurent series of z-z_0 inside each ring where the **function** is *analytic*. ***In other words***,

If a function fails to be analytic at a point z_0 , one can not apply *Taylor's* theorem **at that point**.Unlike the Taylor series which express $f(z)$ as a series of terms with *non-negative powers* of z , a ***Laurent series*** includes terms with *negative powers*.

Theorem

Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$ centered at z_0 and let C denote any poisitive orientated simple closed contour around z_0 and lying in that domain . Then,



$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n + \boxed{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}$$

↑

Principle Part of Laurent's Series

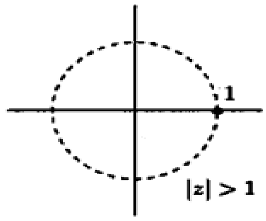
تمثل C_n محيط الدائرة (الخارجية او الداخلية)التي نصف قطرها R_1 و R_2 وكلاهما بالاتجاه الموجب وعكس عقرب الساعة وعلى التوالي .

ملاحظـة :

1. يسمى الجزء $c_0 + c_1(z - z_0) + +c_2(z - z_0)^2 + \cdots$ بالجزء التحليلي من متسلسلة لورنت والجزء الثاني $\frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$ بالجزء الاساسي .
2. اذا كان الجزء الاساس (الرئيسي) من متسلسلة لورنت (صفرا) فان متسلسلة لورنت تصبح " متسلسلة تايلر ".

Example 6 If $|z| > 1$, expand $f(z) = \frac{1}{1-z}$ using Laurent series

Ans. $\frac{1}{1-z} = -\sum_{n=1}^{\infty} \frac{1}{z^n}$, $|z| > 1$



$$f(z) = \frac{1}{1-z} = \frac{1}{z\left(\frac{1}{z}-1\right)} = -\frac{1}{z\left(1-\frac{1}{z}\right)} = -(\frac{1}{z})(\frac{1}{\left(1-\frac{1}{z}\right)})$$

$$\frac{1}{1-z} = -(\frac{1}{z})[1 + \frac{1}{z} + \frac{1}{z^2} = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots.$$

$$\frac{1}{1-z} = -[\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots.]$$

Here $f(z) = \frac{1}{1-z}$ is ana;lytic **everywhere** apart from the **singularity** at $z = 1$.

Above are the expansions of f in the regions inside and outside the circle of radius 1, centered on $z = 0$, where $|z| < 1$ is the region *inside* the circle and $|z| > 1$ is the region *outside* the circle.

Example 7 Expand $\frac{e^{2z}}{(z-1)^3}$; $z = 1$

Solution

Let , $u = z - 1 \rightarrow z = 1 + u$, $2z = 2(1 + u)$

$$\therefore \frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2u}}{u^3} = \frac{e^2}{u^3} \times e^{2u} = \frac{e^2}{u^3}[1 + 2u + \frac{(2u)^2}{2!} + \cdots.$$

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \frac{4e^2(z-1)}{3} + \cdots.$$

حيث ان $z = 1$ هو قطب (pole) من الرتبة الثالثة .

Example 8 Expand $\frac{1}{z^2(z-3)^2}$; $z=3$

Solution

Let , $u=z-3 \rightarrow z=u+3$, $z=u+3$

$$\frac{1}{z^2(z-3)^2}=\frac{1}{u^2(3+u)^2}=\frac{1}{9u^2(1+u/3)^2}$$

$$=\frac{1}{9u^2}\{1+(-2)\left(\frac{u}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^2+\cdots\}$$

$$\therefore \quad \frac{1}{z^2(z-3)^2}=\frac{1}{9u^2}-\frac{2}{27u}+\frac{1}{27}-\frac{4u}{24^3}+\cdots$$

$$\frac{1}{z^2(z-3)^2}=\frac{1}{9(z-3)^2}-\frac{2}{27(z-3)}+\frac{1}{27}-\frac{4u}{24^3}+\cdots$$

حيث ان $z=3$ هو قطب (pole) من الرتبة الثانية .

Example 9 Let $f(z)=\frac{1}{2+z}$ .Determine the Laurent series around $z=1$

Solution

Obviously, we have a simple pole at $z=-2$. Hence, we are dealing with a radius of **3** and want to find the Laurent series for both $|z-1|<3$ *and* $|z-1|>3$. The Laurent series will reduce to a Taylor series inside $|z-1|<3$ where $f(z)$ is analytic.

For $|z-1|<3$, we refer to the well-known geometric series. We begin by trying to create a $(z-1)$ term in the denominator.

$$f(z)=\frac{1}{2+z}=\frac{1}{2+z-1+1}=\frac{1}{3+(z-1)}=\left(\frac{1}{3}\right)\frac{1}{1-\left(\frac{-(z-1)}{3}\right)}$$

Since $\left|\frac{-(z-1)}{3}\right|<1$, we can now represent the function as a series:

$$\Rightarrow \quad f(z)=\frac{1}{3}\sum_{n=0}^{\infty}\frac{-(z-1)^n}{3^n}=\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{3^{n+1}} \quad for \quad |z-1|<3$$

Which is just a Taylor series as the function is analytic inside the region. For $|z-1|>3$, we can use $\frac{3}{|z-1|}$ and follow our previous work to obtain:

$$f(z)=\frac{1}{3+(z-1)}=\frac{1}{z-1}\frac{1}{1-\left(\frac{-3}{z-1}\right)}=\frac{1}{z-1}\sum_{n=0}^{\infty}\frac{(-1)^n3^n}{(z-1)^n}$$

In this case, we have obtained the Laurent expansion. The generalized residue for the outer ring $|z-1| > 3$ is the coefficient of $\frac{1}{z-1}$, that is $b_1 = 1$.

Example 10 Expand $f(z) = \frac{1}{(z+1)(z+3)}$ as Laurent series when:

(a) $1 < |z| < 3$

(b) $|z| < 3$

(c) $0 < |z+1| < 2$

(d) $|z| < 1$

Solution

Using partial fractions

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z-3}$$

$$1 = Az + 3A + Bz + B$$

$$A + B = 0$$

$$3A + B = 1$$

$$A = -B$$

$$3A - A = 1 \rightarrow \quad \boldsymbol{A = 1/2}$$

$$\boldsymbol{B = -1/2}$$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$$

(a) $|z| > 1$ *, then*

$$\frac{1}{2}\left(\frac{1}{z+1}\right) = \frac{1}{2z(1+\frac{1}{z})} = \frac{1}{2z}[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots]$$

$$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

(b) $|z| < 3$ *, then*

$$\frac{1}{2}\left(\frac{1}{z+3}\right) = \frac{1}{6(1+\frac{z}{3})} = \frac{1}{6}[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots]$$

$$\frac{1}{2}\left(\frac{1}{z+3}\right) = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

وعليه فان مفكوك لورنت صحيح لكل من $|z| > 1$, $|z| < 3$ اي ان $|z| > 3$ ينتج بالطرح وهو :

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

(c) $0 < |z+1| < 2$

Let , $u = 1 + z$, then

$$\frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)} = \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right)$$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots$$

$$0 < |z+1| < 2 \quad or\ , \quad |u| < 2 \ , \quad u \neq 0$$

(d) *If* $|z| < 1$ *, then*

$$\frac{1}{2(z+1)} = \frac{1}{2(1+z)} = \frac{1}{2}(1 - z + z^2 - z^3 + \cdots$$

$$= \frac{1}{2} - \frac{1}{2}z + \frac{1}{2}z^2 - \frac{1}{2}z^3 + \cdots$$

اذا كان $|z| < 3$ ، نجد من الفرع (a)

$$\frac{1}{2}\left(\frac{1}{z+3}\right) = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

وبالتالي فان مفكوك لورنت المطلوب والصحيح لكل من $|z| < 1$ ، $|z| < 3$ ، اي ان $|z| < 1$ ينتج بالطرح :

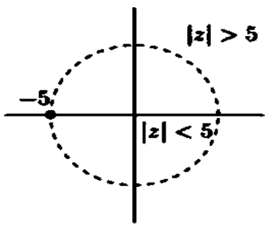
$$= \frac{1}{3} - \frac{4z}{9} + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

وهي متسلسلة تايلر .

Example 11!! Determine the Laurent series for $f(z) = \frac{1}{(z+5)}$ that are valid in the regions**:**

(a) $|z| < 5$

(b) $|z| > 5$



NOTE :

$$f(z) = \frac{1}{5(1+\frac{z}{5})} = \cdots. \quad , \quad f(z) = \frac{1}{z\left(1+\frac{5}{z}\right)} = \cdots.$$

Example 12!! Determine the Laurent series for $f(z) = \frac{1}{(z-i)^2}$ **;** $z_0 = i$

Prof.Dr. Hayfa G. Rashid 2021

Homework

1. Expand the follwing function

(a) $\frac{1}{\sqrt{1+z^3}}$; $z=1$

(b)) $\frac{1}{\sqrt{1+z^3}}$; $z=0$

$\frac{1}{\sqrt{1+z^3}} = 1 - \frac{1}{2}z^3 + \frac{(1)(3)}{(2)(4)}z^6 - \frac{(1)(3)(5)}{(2)(4)(6)}z^9$

(c) $\sin^{-1}z =$

$z + \frac{1}{2}\frac{z^3}{3} + \frac{(1)(3)}{(2)(4)}\frac{z^5}{5} + \frac{(1)(3)(5)}{(2)(4)(6)}\frac{z^7}{7}$; $|z| < 1$

(d) $f(z) = \ln(3 - iz)\,, f(0) = \ln 3$

(e)) $f(z) = \frac{1}{z-3}$ as Laurent series for

(a) $|z| < 3$

(b) $|z| > 3$

Ans.

(a) $\frac{-1}{z} - \frac{z}{9} - \frac{z^2}{9} - \frac{z^3}{81} + ..$

(b) $\frac{-1}{z} + \frac{3}{z^2} + \frac{9}{z^3} + \frac{27}{z^4} + \ldots.$

2.Expand :

$f(z) = \frac{z}{(z-1)(2-z)}$ as Laurent series for ,

(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$, (d) $|z-1| < 1$, (e) $0 < |z-2| < 1$

Ans.

(a) $\frac{-1}{2}z - \frac{3z^2}{4} - \frac{7z^3}{8} - \frac{15z^3}{16} - \ldots.$

(b) $\frac{1}{z^2} + \frac{1}{z} + 1 + \frac{1}{2}z + \frac{1}{4}z^2 + \frac{1}{8}z^3 + \ldots.$

(c) $\frac{-1}{2} - \frac{3}{z^2} - \frac{7}{z^3} - \frac{15}{z^4} + \cdots.$

(d) $\frac{-1}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} + \ldots.$

(e) $1 - \frac{2}{(z-2)} - \frac{1}{(z-2)} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \frac{1}{(z-2)^4} + \ldots.$

3.Expand : $f(z) = \frac{1}{z(z-2)}$; $0 < |z| < 2$, $|z| > 2$

4.Expand : $f(z) = \frac{z}{1+z^2}$; , $|z-3| > 2$

5.Expand : $f(z) = \frac{1}{(z-2)^2}$; $|z| < 2$, $|z| > 2$

6.Expand the following functions as Laurent series around $z = 0$

(a) $f(z) = \frac{1-\cos z}{z}$, [*Ans.* $\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} \ldots$

(b) $f(z)=\frac{e^{z^2}}{z^3}$, [*Ans.* $\frac{1}{z^3}-\frac{1}{z}+\frac{z}{2!}+\frac{z^3}{3!}+\frac{z^5}{4!}+\frac{z^7}{7!}+\cdots$

(c) $f(z)=zsinh\sqrt{z}$, $(d)\ f(z)=ze^z$

7. State the singular points for the functions :

(a) $\frac{1}{2(sinz-1)^2}$, $(b)\ \frac{z}{(e^{1/z}-1)^2}$, $(c)\cos(z^2-z^{-2})$, $(d)\frac{z}{(e^{1/z}-1)}$

$(e)\ tan^{-1}(z^2+2z+2)$, *(f)* $(\frac{z}{(e^z-1)}$