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Complex Integration with Residue

الفصل الثاني

المحاضرة الخامسة

✓ التكامل العقدي باستخدام الرواسب (البواقي) (Residue)

نفرض ان $f(z)$ دالة تحليلية على الدائرة C وداخلها وان z_0 نقطة شاذة معزولة للدالة $f(z)$ ، فان مفكوك متسلسلة لورنت للدالة $f(z)$ هو :

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (1)$$

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots . + \frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \qquad , n = 0, \mp 1, \mp 2, \ldots \qquad (2)$$

عندما كانت $n = -1$ فان المعادلة (2) ،

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

يسمى (a_{-1}) راسب الدالة $f(z)$ عند النقطة z_0 ويرمز له $Res(f, z)$ وعليه فان له $Res(f, z) = a_{-1}$ ، لذلك يكون

$$\oint_C f(z) = 2\pi i a_{-1} = 2\pi i Res(f, z_0)$$

كذلك اذا كان للدالة $f(z)$ قطب من الرتبة n عند النقطة z_0 فان :

$$Res(f, z_0) = \frac{1}{(n-1)!} lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \{f(z)(z - z_0)^n\}$$

Example 1 Evaluate the integral $\oint_C \frac{e^{iz}-sinz}{(z-\pi)^3}dz$ where C is a circle $|z-3|=1$ in positive direction.

Solution

القطب من الرتبة الثالثة $z_0=\pi$

$$Res(f,z_0)=\frac{1}{(n-1)!}lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\{f(z)(z-z_0)^n\}$$

$$Res(f,z_0)=\frac{1}{2}lim_{z\to\pi}\frac{d^2}{dz^2}\left\{(z-\pi)^3\frac{e^{iz}-sinz}{(z-\pi)^3}\right\}$$

$$Res(f,z_0)=\frac{1}{2}lim_{z\to\pi}\frac{d^2}{dz^2}\{e^{iz}-sinz\}=\frac{1}{2}$$

$$\oint_C \frac{e^{iz}-sinz}{(z-\pi)^3}dz=2\pi i\,Res(f,\pi)=2\pi i*\frac{1}{2}=\pi i$$

$$\oint_C \frac{e^{iz}-sinz}{(z-\pi)^3}dz=\pi i$$

Example 2 Use the residue theorem to find $\oint_C \frac{dz}{(z-1)(z+1)}$ where C: $|z|=3$.

Solution

$$z_0=1\ \ ,z_0=-1$$

تمثلان قطبين بسيطين للدالة $f(z)$ ويقعان داخل الدائرة $|z|=3$.

$$f(z)=\frac{1}{(z-1)(z+1)}$$

1. $z_0=1$

$$Res(f,1)=lim_{z\to1}\left\{(z-1)\frac{1}{(z-1)(z+1)}\right\}=\frac{1}{2}$$

2. $z_0=-1$

$$Res(f,-1)=lim_{z\to-1}\left\{(z+1)\frac{1}{(z+1)(z+1)}\right\}=\frac{-1}{2}$$

$$\oint_C \frac{dz}{(z-1)(z+1)}=2\pi i[Res(f,1)+Res(f,-1)]$$

$$\oint_C \frac{dz}{(z-1)(z+1)} = 2\pi i\left[\frac{1}{2} - -\frac{1}{2}\right] = 0$$

$$\oint_C \frac{dz}{(z-1)(z+1)} = 0$$

Example 3 Find the residue of function $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$.

Solution

$$z_0 = -1 \ , \ z_0 = \mp 2i \ , \quad n = 2$$

$z_0 = -1$, قطب من الرتبة الثانية | $z_0 = \mp 2i$, قطبان بسيطان | $n = 2$

1. Residue at $z_0 = -1$

$$Res(f, z_0) = \frac{1}{(n-1)!} lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}\{f(z)(z-z_0)^n\}$$

$$Res(f,-1) = \frac{1}{1!} lim_{z\to -1} \frac{d}{dz}\left\{(z+1)^2 \times \frac{z^2-2z}{(z+1)^2(z^2+4)}\right\}$$

$$es(f,-1) = lim_{z\to -1}\left\{\frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2}\right\} = -\frac{14}{25}$$

2. Residue at $z_0 = 2i$

$$Res(f, z_0) = \frac{1}{(n-1)!} lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}\{f(z)(z-z_0)^n\}$$

$$Res(f,-1) = \frac{1}{1!} lim_{z\to 2i} \frac{d}{dz}\left\{(z-2i) \times \frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\}$$

$$Res(f,-1) = \frac{-4-4i}{(2i+1)^2(4i)} = \frac{7+i}{25}$$

3. Residue at $z_0 = -2i$

$$Res(f,-1) = \frac{1}{1!} lim_{z\to -2i} \frac{d}{dz}\left\{(z+2i) \times \frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\}$$

$$Res(f,-1) = \frac{-4+4i}{(-2i+1)^2(-4i)} = \frac{7-i}{25}$$

Example 4 Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz$ where C is a circle $|z| = 3$.

Solution

$$f(z) = \frac{e^{zt}}{z^2(z^2+2z+2)}$$

$$z_0 = 0, \quad z_0 = -1 \mp i\ , \quad n = 2$$

قطب من الرتبة الثانية

1. $$Res(f,-1) = \frac{1}{1!} lim_{z\to 0} \frac{d}{dz}\left\{z^2 \times \frac{e^{zt}}{z^2(z^2+2z+2)}\right\}$$

$$= lim_{z\to 0}\left\{\frac{(z^2+2z+2)(te^{zt}) - (e^{zt})(2z+2)}{z^2(z^2+2z+2)}\right\}$$

$$Res(f,-1) = \frac{t-1}{2}$$

2. Residue at $z_0 = -1+i$

$$Res(f,-1+i) = lim_{z\to -1+i}\left\{\{z-(-1+i)\} \times \frac{e^{zt}}{z^2\{z-(-1+i)\}\{z-(-1-i)\}}\right\}$$

$$Res(f,-1+i) = \frac{e^{(-1+i)t}}{(-1+i)^2} \times \frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

3. Residue at $z_0 = -1-i$

$$Res(f,-1+i) = lim_{z\to -1-i}\left\{\{z-(-1-i)\} \times \frac{e^{zt}}{z^2\{z-(-1-i)\}\{z-(-1+i)\}}\right\}$$

$$Res(f,-1+i)=\frac{e^{(-1-i)t}}{(-2i)(-1-i)^2}=\frac{e^{(-1-i)t}}{4}$$

مجموع الرواسب ،

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz=2\pi i$$

$$=2\pi i\left[\frac{t-1}{2}+\frac{e^{(-1+i)t}}{4}+\frac{e^{(-1-i)t}}{4}\right]$$

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz=2\pi i\left[\frac{t-1}{2}+\frac{e^{-t}}{2}cost\right]$$

اي ان ،

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz=\frac{1}{2}(t-1)+\frac{1}{2}e^{-t}cost$$