Chapter Five
Boolean Ring and Extension Field

Definition :- **(Boolean Ring حلقة بوليين)**

A **Boolean ring** $(R , + , .)$ is a ring with identity and every element in a ring R is an idempotent element , That is $a^2 = a , \forall a \in R$.

Example :-

A ring $(Z_2 , +_2 , ._2)$ is a Boolean ring . Since $(Z_2 , +_2 , ._2)$ is a ring with identity and, $\forall a$ in a ring $Z_2 = \{ \bar{0} , \bar{1} \}$ is an idempotent element

$\overset{(i.e)}{\Longrightarrow} \bar{0}^2 = \bar{0} ._2 \bar{0} = \bar{0}$ and $\bar{1}^2 = \bar{1} ._2 \bar{1} = \bar{1}$.

Example(2):-

A ring $(P(X), \Delta , \cap)$, Where $P(X)=\{A : A \subseteq X \}$ is a Boolean ring .

Since $(P(X), \Delta , \cap)$ is a ring with identity and $\forall A \in P(X) \Longrightarrow A^2 = A \cap A = A$

Example (3) :-

A ring $(Z_3 , +_3 , ._3)$ is not Boolean ring . Since $\exists \bar{2} \in Z_3$ is not idempotent element , s .t $\bar{2}^2 = \bar{2} ._3 \bar{2} = \bar{1} \neq \bar{2}$.

Theorem (5-1):- Prove that every Boolean ring $(R, +, .)$ is a commutative ring with the characteristic 2 .

Proof :- المعطى R is a Boolean ring

< First to prove a Boolean ring R is a comm. ring >?

Let $a, b \in R$, since R is a ring $\Longrightarrow a + b \in R$

$\Longrightarrow a + b = (a + b)^2$ (Since R is a Boolean ring)

$\Longrightarrow a + b = (a + b).(a + b)$

$\Longrightarrow a + b = a^2 + a.b + b.a + b^2$

$\Longrightarrow \cancel{a} + \cancel{b} = \cancel{a} + a.b + b.a + \cancel{b}$ $(\because a^2 = a \text{ and } b^2 = b)$

$\Longrightarrow 0 = a.b + b.a$

$\Longrightarrow a.b = - b.a$

$\because b \in R \Longrightarrow -b \in R$ and R is a Boolean ring

$\Longrightarrow -b = (-b)^2 = b^2 = b \Longrightarrow a.b = - b.a = b.a$

Therefore, R is a commutative ring .

< Now, to prove , Ch(R) = 2 >?

< (i.e) to prove , $\forall a \in R \Longrightarrow 2a = 0$ >?

Let $a \in R$, and since R is a ring $\Longrightarrow a + a \in R$

$\Longrightarrow a + a = (a + a)^2$ (Since R is a Boolean ring)

$\Longrightarrow a + a = a^2 + 2a + a^2$

$\Longrightarrow \cancel{a} + \cancel{a} = \cancel{a} + 2a + \cancel{a}$ $(\because a \in R \Longrightarrow a^2 = a)$

$\Longrightarrow 0 = 2a$

$\Longrightarrow 2a = 0 , \forall a \in R$

Thus , Ch(R) = 2 . □

Theorem (5-2):- If (R , + , .) is a Boolean ring . Then (R/I , + , .) is also Boolean ring .

Proof :- المعطى **R is a Boolean ring**

< T.P. R/I is a Boolean ring >?

$\because$ R is a Boolean ring

$\Longrightarrow$ R is a ring with identity

$\Longrightarrow$ by Corollary of Theorem (2-15) we get , R/I is also ring with identity $\cdots *$

Let $a + \mathrm{I}$ in a ring R/I

$\therefore$ $(a + \mathrm{I})^2 = (a + \mathrm{I}).(a + \mathrm{I}) = a^2 += a + \mathrm{I}$

$\Longrightarrow \forall$ element $a + \mathrm{I}$ in a ring R/I is an idempotent element $\cdots **$

Therefore , by $*$ and $**$ we get , a ring (R/I , + , .) is a Boolean ring

Theorem (5-3):-

If (R , + , .) is a Boolean ring and (I , + , .) is a proper ideal of a ring R Then , I is a maximal ideal iff I is a prime ideal .

Theorem (5-4):-

A Boolean ring (R,+, .)is a field if and only if (R , + , .) $\simeq$ (Z_2 , $+_2$, $\cdot_2$)

Theorem (5-5):- An proper ideal (I , + , .) in a Boolean ring (R , + , .) is a maximal ideal iff (R/I , + , .) $\simeq$ (Z_2 , $+_2$, $\cdot_2$) . البرهان واجب

Extension Field

Definition :- (**Extension Field** توسيع الحقل)

Let (F , + , .) be a subfield of a field (E , + , .) . Then E is called an **extension field** of F .

Example (1):-

A field (₵ , + , .) is an extension field of ($\mathcal{R}$, + , .) . Since ($\mathcal{R}$, + , .) is a subfield of a field(₵ , + , .) , s.t

1- $a - b \in \mathcal{R}$, $\forall\, a, b \in \mathcal{R}$
2- $a \cdot b^{-1} \in \mathcal{R}$, $\forall\, a, b \in \mathcal{R}$ and $b \neq 0$

Example (2):-

A field ($\mathcal{R}$, + , .) is an extension field of (Q , + , .) . Since (Q, + , .) is a subfield of a field($\mathcal{R}$, + , .) , s.t

1- $a - b \in Q$, $\forall\, a, b \in Q$
2- $a \cdot b^{-1} \in Q$, $\forall\, a, b \in Q$ and $b \neq 0$

التالي تعريف أخر للتوسيع الحقول :

Definition :- (**Extension Field** توسيع الحقل)

Let (F , + , .) be a subfield of a field (E , + , .) . If there exists a polynomial f(x)$\in$ F has no roots in F , but f(x) has roots in a field E . Then E is called an **extension field** of F .

Example (3):- A field (₵ , + , .) is an extension field of ($\mathcal{R}$, + , .) . Since $\exists$ $f(x) = x^2 + 1 \in \mathcal{R}$ has no roots in $\mathcal{R}$. But f(x) has roots in ₵ . where the roots of f(x) are $x = \pm i \in$ ₵ , but $x = \pm i \notin \mathcal{R}$.

Example (4):- A field $(\mathcal{R}, +, .)$ is an extension field of $(Q, +, .)$. Since $\exists$ $f(x) = x^2 - 2 \in Q$ has no roots in Q . But f(x) has roots in $\mathcal{R}$. where the roots of f(x) are $x = \pm\sqrt{2} \in \mathcal{R}$, but $x = \pm\sqrt{2} \notin Q$.